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Research article

A generating function framework for the no-feedback card guessing game after riffle shuffles

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Abstract: We introduce a generating-function framework for analyzing the no-feedback card-guessing game after k Gilbert–Shannon–Reeds riffle shuffles. We show that the distribution of the card appearing in position i can be expressed as a structured mixture of 2^k tractable components, each corresponding to a sum of independent Bernoulli trials. From this decomposition, we derive an explicit closed-form expression for the probability generating function, represented as a product of binomial-type polynomials with a clear and systematic structure, valid for any number of cards n and any number of shuffles k. This formulation replaces recursive convolutions with a single analytic expression, enabling efficient computation and revealing the combinatorial–probabilistic structure underlying riffle shuffles. Beyond exact evaluation, the framework connects optimal no-feedback strategies with the generating functions and suggests asymptotic behavior in both the fixed-k, large-n and fixed-n, large-k regimes.

Keywords: card guessing; no-feedback strategy; Gilbert–Shannon–Reeds riffle shuffle; probability generating function; mixture distribution
Mathematics Subject Classification: 05A15, 60C05

1. Background and formulation

Over the past few decades, the card guessing game has been the subject of extensive study, with attention given to optimal strategies, worst-case performance, and the distribution of correct guesses [1, 2]. A fundamental feature of the game is the use of riffle shuffles, which are widely employed in practical settings, such as casinos and poker games, where multiple iterations are used to approximate randomness. The classical result of Bayer and Diaconis [3] established that after seven riffle shuffles, the positions of all 52 cards are nearly uniformly distributed.

Throughout this work we adopt the Gilbert–Shannon–Reeds (GSR) model of riffle shuffling, in which the deck is cut into two piles and then interleaved with probabilities proportional to pile sizes [3]. This is the standard probabilistic model of riffle shuffles used in modern analyses. The game admits several variants, depending on whether feedback is provided by the dealer. In the complete feedback version, analogous to blackjack, the dealer reveals each card after a guess, allowing the player to adapt their strategy dynamically. In the no-feedback version, which is the focus of this work, the player must commit to a complete sequence of guesses in advance. Repetition of guesses across different positions is permitted to increase the likelihood of correct matches.

The variant studied in this paper is summarized as follows:

Objective:

`Maximize the number of correct guesses of the card values.`

Rules:

- The game begins with a deck of n cards, initially ordered as $1, 2, 3, \ldots, n$.
- The dealer performs k riffle shuffles.
- The player then makes one guess for each position in the deck.
- No feedback is provided during or after the guessing phase.
- The game concludes after all n positions have been guessed.

In this no–feedback setting, the expected number of correct guesses is the sum of the per–position probabilities of a correct match. By linearity of expectation, these terms can be maximized independently, so the optimal strategy is simply to guess, at each position, the card value with the highest marginal probability. The central contribution of this paper is to provide explicit formulas for these probabilities through a generating function decomposition.

Our contribution: Extending beyond prior literature

Related work on the complete-feedback version of the game was given by Krityakierne and Thanatipanonda [4], who developed a generating-function framework for analyzing the moments of correct guesses after a single riffle shuffle. In 1998, Ciucu [5] analyzed the no-feedback version of the card-guessing game under riffle shuffles for large k, establishing the optimal strategy when $k > 2\log_2(n)$ via spectral methods on the transition matrix. Subsequent work continued along this line. Krityakierne et al. [6] analyzed the one–shuffle, no–feedback game under Ciucu's one–shuffle optimal strategy, constructing counting generating functions with efficient recurrences and deriving an exact closed form for the expected number of correct guesses together with higher–moment formulas. Recently, Kuba and Panholzer [7], building on the work of Krityakierne et al. [6], established a limit law for the number of correct guesses in the one-shuffle, no-feedback setting.

In contrast, the present paper introduces a new generating-function framework that applies to arbitrary n and any number k of riffle shuffles in the no-feedback setting. Whereas previous studies used generating functions to derive moments of the number of correct guesses [4, 6], our approach reveals the underlying combinatorial–probabilistic structure of the position distribution and unifies the shuffle process within a single analytic formulation. To our knowledge, it provides the first comprehensive description of the position distribution after finitely many shuffles, yielding both exact formulas and asymptotic insight. In a broader context, generating functions are classical tools in enumerative combinatorics and discrete probability [8, 9], and our formulation leverages their expressive power to connect combinatorial structure with probabilistic behavior. This formulation not only clarifies the underlying structure of riffle shuffles but also leads to concrete applications, including asymptotically optimal guessing strategies and connections with classical mixing-threshold results.

2. Preliminaries: Single riffle shuffle ($k = 1$)

Consider a single riffle shuffle applied to a deck of n cards. In the GSR model [3], the deck is first cut into two packets according to the binomial distribution, and then the packets are interleaved by sequentially dropping the next card from either packet with probabilities proportional to the packet sizes. An immediate question arises: Which permutations can result from such a shuffle and with what probabilities?

For example, when $n = 1$, the only resulting permutation is $\{1\}$, but it occurs with multiplicity 2, corresponding to the interleavings of $(\{1\}, \emptyset)$ and $(\emptyset, \{1\})$. In general, there are 2^n possible outcomes, among which $n + 1$ are identity permutations (those with a single increasing subsequence), while the remaining permutations each exhibit two increasing subsequences and occur with multiplicity one. Several examples are visualized in Figure 1, where the color of each cell indicates the most likely card value at the corresponding position.

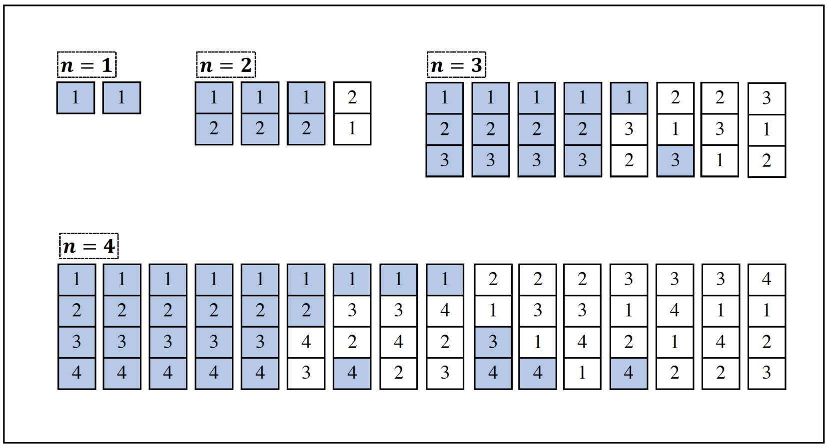


Figure 1. All possible permutations obtained from a single riffle shuffle for $n = 1, 2, 3, 4$. For each position, the color encodes the most likely card value.

2.1. *Frequency and probability matrices*

Let $M = (m_{i,j})$ denote the frequency matrix, where $m_{i,j}$ represents the number of occurrences of card j appearing in position i across all 2^n possible shuffles. Each row i corresponds to a deck position, while each column j indexes the card value.

For instance, when $n = 4$, the matrix M is given by

$$M = \begin{bmatrix} \boxed{\mathbf{9}} & 3 & 3 & 1 \\ 4 & \boxed{\mathbf{6}} & 4 & 2 \\ 2 & 4 & \boxed{\mathbf{6}} & 4 \\ 1 & 3 & 3 & \boxed{\mathbf{9}} \end{bmatrix}.$$

2.2. *Probability matrix for $k = 1$*

Define the probability matrix $P^{(1)} := M/2^n$, whose entries are denoted $a(i, j, n) := m_{i,j}/2^n$. Then, $a(i, j, n)$ represents the probability that card j appears in position i after a single riffle shuffle. It is known that this quantity admits the following closed-form expression:

$$a(i, j, n) = \frac{1}{2^i}\binom{i-1}{j-1} + \frac{1}{2^{n-i+1}}\binom{n-i}{j-i}. \tag{2.1}$$

This identity was established in [5]. Note that the first term is nonzero when $i \geq j$, while the second term is nonzero when $j \geq i$.

The matrix $P^{(1)}$ is doubly stochastic: the sums of each row and each column equal 1. The row sums follow directly from the binomial theorem, since

$$\sum_{j=1}^{n} a(i, j, n) = \frac{1}{2^i}\sum_{j=1}^{i}\binom{i-1}{j-1} + \frac{1}{2^{n-i+1}}\sum_{j=i}^{n}\binom{n-i}{j-i} = \frac{1}{2} + \frac{1}{2} = 1.$$

The column sums are less immediate, but also satisfy

$$\sum_{i=1}^{n} a(i, j, n) = \sum_{i=j}^{n}\frac{1}{2^i}\binom{i-1}{j-1} + \sum_{i=1}^{j}\frac{1}{2^{n-i+1}}\binom{n-i}{j-i} = 1.$$

To establish this, we apply induction on j and n, with $j \leq n$, with base cases $j = 1$ and $j = n$.

Additionally, $P^{(1)}$ exhibits a symmetry property:

$$a(i, j, n) = a(n + 1 - i, n + 1 - j, n), \tag{2.2}$$

a fact already observed in Ciucu [5].

2.3. *Probability matrix for general k-shuffle: $P^{(k)}$*

We now consider the case where the riffle shuffle is applied k times. Let $a_{i,j}^{(k)}$ denote the probability that card j occupies position i after k independent riffle shuffles. We define the matrix $P^{(k)} = \left(a_{i,j}^{(k)}\right)$.

As noted by Ciucu [5], the transpose $\left(P^{(1)}\right)^{\intercal}$ acts as a transition matrix for the Markov chain on permutations induced by the shuffle. Therefore, the matrix power relation holds:

$$a_{i,j}^{(k)} = \sum_{s_1=1}^{n} a_{i,s_1}^{(1)} a_{s_1,j}^{(k-1)} = \cdots = \sum_{s_1=1}^{n} \cdots \sum_{s_{k-1}=1}^{n} a_{i,s_1}^{(1)} a_{s_1,s_2}^{(1)} \cdots a_{s_{k-1},j}^{(1)}. \tag{2.3}$$

This recursive structure underlies the key decomposition and generating function representations developed in the subsequent sections.

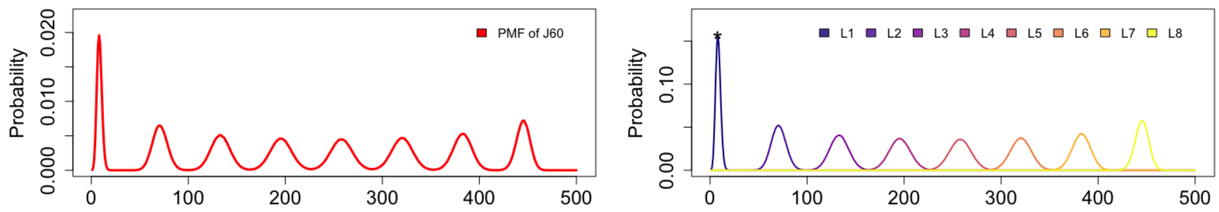
3. Main results

3.1. Mixture decomposition and generating functions for J_i

Fix a position $i \in \{1, \dots, n\}$, and consider the i-th row of the probability matrix $P^{(k)}$. We define the discrete random variable J_i, representing the card number appearing in position i after k riffle shuffles. Its probability mass function is given by

$$\mathbb{P}(J_i = j) = a_{i,j}^{(k)}, \quad j = 1, \dots, n.$$

Figure 2 (left) shows the p.m.f. of J_{60} for $n = 500$ and $k = 3$. The shape of the distribution suggests an underlying mixture structure. This observation motivates a decomposition of the p.m.f. into structured components, which we now formalize.



(a) The original mixture distribution (b) The decomposed components of J60

Figure 2. Example with $k = 3$, $n = 500$, $i = 60$: (a) the p.m.f. of J_{60}, that is, $\left(a_{60,j}^{(3)}\right)_{j=1}^{n}$; (b) the mixture components p_L of the decomposition of J_{60}.

Decomposability of $a_{i,j}^{(k)}$

We begin with the case $k = 1$, where the transition probability $a_{i,j}^{(1)}$ admits a closed-form expression that splits naturally into two terms:

$$a_{i,j}^{(1)} = \frac{1}{2}\left[b_{i,j}^{(1)} + b_{i,j}^{(2)}\right], \tag{3.1}$$

where

$$b_{i,j}^{(1)} := \frac{1}{2^{i-1}}\binom{i-1}{j-1}, \qquad b_{i,j}^{(2)} := \frac{1}{2^{n-i}}\binom{n-i}{j-i}. \tag{3.2}$$

For general $k \geq 1$, the term $a_{i,j}^{(k)}$ can be recursively decomposed into a mixture of 2^k components, each corresponding to a binary path through a sequence of $b^{(1)}$ and $b^{(2)}$ matrices.

Proposition 3.1 (Mixture decomposition of J_i)**.** *For any $k \geq 1$, the expression for $a_{i,j}^{(k)}$ in* (2.3) *satisfies*

$$a_{i,j}^{(k)} = \frac{1}{2^k}\sum_{L} p_L(j), \tag{3.3}$$

where the sum is over all binary lists $L = (L[1], \dots, L[k]) \in \{1, 2\}^k$. Each component $p_L(j)$ is defined by

$$p_L(j) := \sum_{s_1=1}^{n} \cdots \sum_{s_{k-1}=1}^{n} b_{i,s_1}^{(L[1])} b_{s_1,s_2}^{(L[2])} \cdots b_{s_{k-1},j}^{(L[k])}. \tag{3.4}$$

Proof. Immediate by recursively applying the base decomposition (3.1) within the convolution structure of (2.3). □

Examples

When $k = 2$, the mixture consists of four components:

$$a_{i,j}^{(2)} = \frac{1}{4}\left[p_{[1,1]}(j) + p_{[2,1]}(j) + p_{[1,2]}(j) + p_{[2,2]}(j)\right],$$

where each component is expressed as

$$\begin{aligned}
p_{[1,1]}(j) &= \sum_{s=1}^{n} b_{i,s}^{(1)} b_{s,j}^{(1)} = \sum_{s=1}^{n} \frac{1}{2^{i-1}}\binom{i-1}{s-1}\frac{1}{2^{s-1}}\binom{s-1}{j-1},\\
p_{[2,1]}(j) &= \sum_{s=1}^{n} b_{i,s}^{(2)} b_{s,j}^{(1)} = \sum_{s=1}^{n} \frac{1}{2^{n-i}}\binom{n-i}{s-i}\frac{1}{2^{s-1}}\binom{s-1}{j-1},\\
p_{[1,2]}(j) &= \sum_{s=1}^{n} b_{i,s}^{(1)} b_{s,j}^{(2)} = \sum_{s=1}^{n} \frac{1}{2^{i-1}}\binom{i-1}{s-1}\frac{1}{2^{n-s}}\binom{n-s}{j-s},\\
p_{[2,2]}(j) &= \sum_{s=1}^{n} b_{i,s}^{(2)} b_{s,j}^{(2)} = \sum_{s=1}^{n} \frac{1}{2^{n-i}}\binom{n-i}{s-i}\frac{1}{2^{n-s}}\binom{n-s}{j-s}.
\end{aligned}$$

Likewise, for $k = 3$, a sample list $L = [1, 1, 2]$ gives

$$p_{[1,1,2]}(j) = \sum_{s_1=1}^{n}\sum_{s_2=1}^{n} b_{i,s_1}^{(1)} b_{s_1,s_2}^{(1)} b_{s_2,j}^{(2)} = \sum_{s_1=1}^{n}\sum_{s_2=1}^{n} \frac{1}{2^{i-1}}\binom{i-1}{s_1-1}\frac{1}{2^{s_1-1}}\binom{s_1-1}{s_2-1}\frac{1}{2^{n-s_2}}\binom{n-s_2}{j-s_2}.$$

Visualization of decomposition

The right panel of Figure 2 displays the individual components p_L of the mixture decomposition for the case $k = 3$, $n = 500$, and $i = 60$. Each curve corresponds to one of the $2^3 = 8$ possible binary list indices $L \in \{1, 2\}^3$, showing the contribution of that component to the total distribution of J_{60}. These components exhibit distinct shapes and modes, further highlighting the structured nature of the decomposition and motivating the generating function approach that follows.

Generating functions for each component p_L

To handle the summation structure more systematically, we define the probability generating function (p.g.f.) of each p_L as

$$F_L(x) := \sum_{j=1}^{n} p_L(j)\, x^j = \sum_{j=1}^{n}\sum_{s_1=1}^{n}\cdots\sum_{s_{k-1}=1}^{n} b_{i,s_1}^{(L[1])} b_{s_1,s_2}^{(L[2])} \cdots b_{s_{k-1},j}^{(L[k])} \cdot x^j. \tag{3.5}$$

Proposition 3.2. *For each list $L \in \{1, 2\}^k$, the function $F_L(x)$ is a valid probability generating function.*

Proof. By construction, the coefficients $p_L(j)$ are non-negative and sum to 1. Hence, $F_L(1) = 1$ and the result follows. □

3.2. *Structured form of $F_L(x)$ via generating functions*

We now examine the structure of the generating function $F_L(x)$ defined in (3.5), by reorganizing the summation (i.e., moving the summation over j innermost) and evaluating the sums sequentially from the inside out. This process yields a compact, tractable expression for $F_L(x)$ for all $k \geq 1$. We begin with illustrative examples.

Examples

- For $k = 0$,
$$F_{[\,]}(x) = x^i.$$
This reflects the trivial case with no shuffling: the card in position i must be card i with probability 1.

- For $k = 1$,
$$F_{[1]}(x) = (0x+1)^{n-i}\left(\frac{x}{2}+\frac{1}{2}\right)^{i-1} x, \quad F_{[2]}(x) = \left(\frac{x}{2}+\frac{1}{2}\right)^{n-i}(x+0)^{i-1}x.$$

- For $k = 2$,
$$F_{[1,1]}(x) = (0x+1)^{n-i}\left(\frac{x}{4}+\frac{3}{4}\right)^{i-1} x, \qquad F_{[2,1]}(x) = \left(\frac{x}{4}+\frac{3}{4}\right)^{n-i}\left(\frac{x}{2}+\frac{1}{2}\right)^{i-1} x,$$
$$F_{[1,2]}(x) = \left(\frac{x}{2}+\frac{1}{2}\right)^{n-i}\left(\frac{3x}{4}+\frac{1}{4}\right)^{i-1} x, \qquad F_{[2,2]}(x) = \left(\frac{3x}{4}+\frac{1}{4}\right)^{n-i}(x+0)^{i-1}\, x.$$

To encapsulate the general form, we define the following notation:
$$g(a,b) := [ax + (1-a)]^{n-i} \cdot [bx + (1-b)]^{i-1} \cdot x. \tag{3.6}$$

Remark 3.1. The function $g(a, b)$ is the probability generating function of the random variable
$$1 + \sum_{j=1}^{n-i} X_j + \sum_{j=1}^{i-1} Y_j,$$
where $X_j \sim \mathrm{Bern}(a)$, $Y_j \sim \mathrm{Bern}(b)$, and all variables are independent.

With this notation, the expressions above become
$$F_{[\,]}(x) = g(0,1), \quad F_{[1]}(x) = g\left(0, \tfrac{1}{2}\right), \quad F_{[2]}(x) = g\left(\tfrac{1}{2}, 1\right).$$

Recursive structure and index ordering

The bisection tree in Figure 3 encodes the recursive structure of the list indices L_t and their associated generating functions. The function $F_{L_t}(x)$ for a given k is defined recursively by splitting the arguments of $g(a, b)$ at each step:
$$g(a,b) \mapsto \begin{cases} g\left(a, \frac{a+b}{2}\right), & \text{left child,} \\ g\left(\frac{a+b}{2}, b\right), & \text{right child.} \end{cases}$$

This tree induces a natural indexing $t \mapsto L_t$, which we use in the statement of the main theorem.

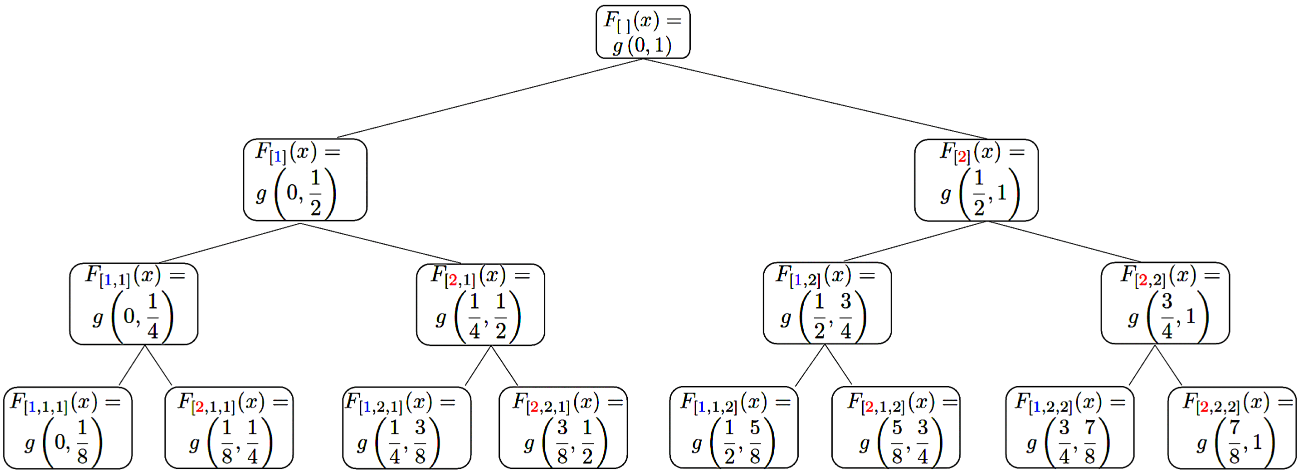


Figure 3. A bisection tree structure. Each node at depth k corresponds to the function $F_{L_t}(x)$ for $t = 0, 1, \ldots, 2^k - 1$.

Theorem 3.3 (Generating function $F_{L_t}(x)$)**.** *For any $k \geq 1$, and list indices $L_0, L_1, \ldots, L_{2^k-1}$ defined via the bisection tree construction, we have*

$$F_{L_t}(x) = g\left(\frac{t}{2^k}, \frac{t+1}{2^k}\right), \quad \textit{for } t = 0, 1, \ldots, 2^k - 1.$$

To prove the theorem, we require the following lemma.

Lemma 3.4. *Assume $f(i) = A^{n-i}B^{i-1}x$, where $A = ax + (1-a)$ and $B = bx + (1-b)$. Then,*

$$T_1(f) := \sum_{s=1}^{i} \frac{1}{2^{i-1}} \binom{i-1}{s-1} f(s) = A^{n-i} \left(\frac{A+B}{2}\right)^{i-1} x;$$

$$T_2(f) := \sum_{s=i}^{n} \frac{1}{2^{n-i}} \binom{n-i}{s-i} f(s) = \left(\frac{A+B}{2}\right)^{n-i} B^{i-1} x.$$

Proof. We first recall two identities derived from the binomial theorem:

$$\sum_{s=1}^{i} \frac{1}{2^{i-1}} \binom{i-1}{s-1} x^{s-1} = \left(\frac{x}{2} + \frac{1}{2}\right)^{i-1},$$

and

$$\sum_{s=i}^{n} \frac{1}{2^{n-i}} \binom{n-i}{s-i} x^{s-1} = \left(\frac{x}{2} + \frac{1}{2}\right)^{n-i} x^{i-1}.$$

We now proceed with evaluating $T_1(f)$ using the definition $f(s) = A^{n-s}B^{s-1}x$. Rewriting,

$$T_1(f) = \sum_{s=1}^{i} \frac{1}{2^{i-1}} \binom{i-1}{s-1} A^{n-s}B^{s-1}x = A^{n-1}x \sum_{s=1}^{i} \frac{1}{2^{i-1}} \binom{i-1}{s-1} \left(\frac{B}{A}\right)^{s-1}$$

$$= A^{n-1}x\left(\frac{\frac{B}{A}+1}{2}\right)^{i-1} \qquad \text{(by binomial identity above)}$$
$$= A^{n-1}x\left(\frac{A+B}{2A}\right)^{i-1} = A^{n-i}\left(\frac{A+B}{2}\right)^{i-1} x.$$

The proof of the claim for $T_2(f)$ follows by an analogous argument. □

Proof of Theorem 3.3. The proof proceeds by induction on $k = |L|$. The base case $k = 0$ is clear, since $F_{[\,]}(x) = x^i = g(0, 1)$.

For the inductive step, suppose $F_{L'}(x) = g\left(\frac{t}{2^{k-1}}, \frac{t+1}{2^{k-1}}\right)$, and let $L = [l_k, L']$. By Lemma 3.4, if $l_k = 1$, then

$$F_L(x) = T_1\left(F_{L'}(x)\right) = g\left(\frac{2t}{2^k}, \frac{2t+1}{2^k}\right),$$

and if $l_k = 2$, then

$$F_L(x) = T_2\left(F_{L'}(x)\right) = g\left(\frac{2t+1}{2^k}, \frac{2t+2}{2^k}\right).$$

□

Remark 3.2. While a complete combinatorial interpretation of $F_L(x)$ remains open, its algebraic form suggests that it corresponds to the p.g.f. of a shifted sum of independent Bernoulli trials, partitioned across two blocks. Each coefficient of x^j corresponds to the total weight of choosing $j-1$ successes under a specific interleaving structure indexed by L. Identifying a concrete combinatorial object or bijection that captures this behavior is an open problem.

Corollary 3.5. *Fix i, n, k. Let J_i denote the random variable representing the card in position i after k riffle shuffles. Define its probability generating function as*

$$G_i(x) = \sum_{j=1}^{n} P(J_i = j)x^j.$$

Then, the p.g.f. of J_i is the sum of the product of polynomials given by

$$G_i(x) = \frac{1}{2^k}\sum_{t=0}^{2^k-1}\left(\frac{t}{2^k}x + 1 - \frac{t}{2^k}\right)^{n-i}\left(\frac{t+1}{2^k}x + 1 - \frac{t+1}{2^k}\right)^{i-1} x. \tag{3.7}$$

Proof.

$$G_i(x) = \sum_{j=1}^{n} a_{i,j}^{(k)} x^j = \frac{1}{2^k}\sum_{t=0}^{2^k-1}\sum_{j=1}^{n} p_{L_t}(j)x^j = \frac{1}{2^k}\sum_{t=0}^{2^k-1} F_{L_t}(x).$$

The second equality follows from (3.3), and the result is a consequence of Theorem 3.3. □

4. Conclusions, applications and outlook

Implications for card guessing

We developed a generating-function framework for analyzing the no-feedback card-guessing game after k riffle shuffles. By decomposing the marginal distribution of card positions into a mixture of 2^k structured components, we obtained an explicit and interpretable probability generating function

$$G_i(x) = \frac{1}{2^k}\sum_{t=0}^{2^k-1}\left(\tfrac{t}{2^k}x + 1 - \tfrac{t}{2^k}\right)^{n-i}\left(\tfrac{t+1}{2^k}x + 1 - \tfrac{t+1}{2^k}\right)^{i-1}x,$$

thereby replacing the $(k-1)$-fold convolution in (2.3) with a single summation. Each summand is the p.g.f. of a sum of independent Bernoulli trials (Remark 3.1), so that $\Pr(J_i = j)$—the probability that card number j occupies position i—is an average of 2^k explicitly parameterized component distributions.

For fixed k and large n, Figure 4 shows that as n increases from 50 to 400, the component distributions separate more clearly. In particular, for large n, $\Pr(J_i = j)$—the coefficient of x^j in $G_i(x)$—is dominated by a single summand rather than by the full mixture. This observation motivates Conjecture 4.1, providing a tractable asymptotic description in the regime of fixed k and large n (see Application A.1). By contrast, when k is large compared to n, the averaging structure leads to behavior consistent with the mixing threshold established in Ciucu's theorem (see Application A.2).

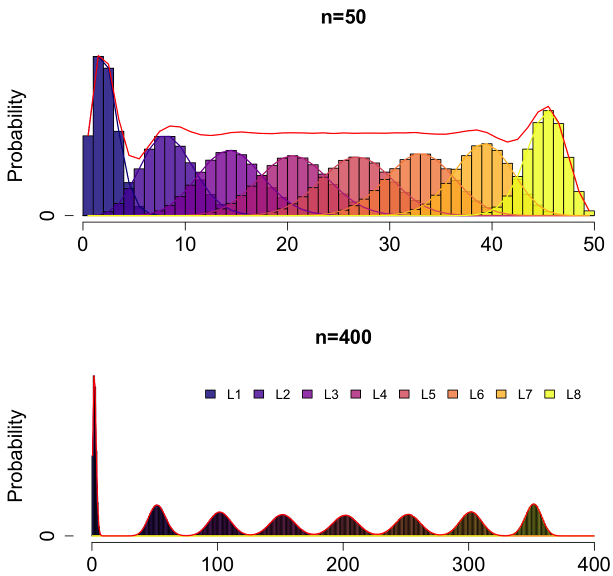


Figure 4. Overlay of the eight component p.m.f.s for $k = 3$ and $i = 15$, with $n = 50$ (top) and $n = 400$ (bottom). The red curve shows the sum of the 2^3 components of $G_i(x)$, whose height at position j equals 2^3 times the coefficient of x^j.

A.1 Asymptotically optimal guessing for fixed k, large n. When n is large relative to k, our explicit form of $G_i(x)$ offers a route toward establishing the optimal no-feedback guessing strategy.

Conjecture 4.1 (Asymptotically optimal guessing for fixed k, large n)**.** *For any fixed $k \geq 1$, there exists $n(k)$ such that for all $n \geq n(k)$, the optimal no-feedback guess j_i^* at position $i \in \{1, \ldots, n\}$ is*

$$\begin{aligned} \textit{if } i \leq \left\lfloor \tfrac{n+1}{2} \right\rfloor: \quad j_i^* &= \left\lfloor \tfrac{i}{2^k} \right\rfloor + 1, \\ \textit{if } i \geq \left\lceil \tfrac{n+1}{2} \right\rceil: \quad j_i^* &= \left\lfloor (n-i+1)\Big(1 - \tfrac{1}{2^k}\Big) \right\rfloor + i. \end{aligned}$$

Supporting rationale. Based on our numerical results, for each i, $\Pr(J_i = j)$ is sharply peaked around a dominant Bernoulli-sum component with parameters $(\frac{t}{2^k}, \frac{t+1}{2^k})$. Maximizing this term yields the offsets in Conjecture 4.1, which remain consistent across all i for large n. A rigorous proof is left as an open problem.

A.2 Asymptotically optimal guessing for fixed n, large k. The mixture structure suggests an alternative proof of Ciucu's theorem.

Theorem 1.1 (Ciucu [5]). For $k \geq 2\log_2(2n) + 1$, the optimal no-feedback guessing strategy after k riffle shuffles of a deck of $2n$ cards is to guess 1 at the first n positions and $2n$ at the remaining n positions.

Connections to existing work on riffle shuffles and dynamical structure

Riffle shuffling is closely connected to the theory of carries and to Holte's *Amazing Matrix* [10], building on earlier work by Diaconis, McGrath, and Pitman [11] and later developed by Diaconis and Fulman [12]. In the GSR model of a b-shuffle, the deck is cut into b packets according to a multinomial law and then interleaved at random. Holte's matrix P_b governs descent statistics of a permutation after one b-shuffle, while another standard matrix Q_b tracks the position of a single marked card. In the case $b = 2$, our probability matrix for one shuffle (2.1) satisfies

$$P^{(1)}(i, j) = Q_2(j, i),$$

so that $P^{(1)} = Q_2^{\intercal}$, and hence $P^{(k)} = (Q_2^k)^{\intercal}$. Thus, $P^{(k)}$ and Q_2^k share the same spectrum. Although one can exploit the diagonalization of Q_2 to analyze $P^{(k)}$, the resulting formulas are less transparent than the product-form generating functions of Corollary 3.5, which recast the shuffle process as a simple mixture structure and make both exact computation and asymptotic analysis more accessible. The resulting 2^k-component decomposition mirrors the binary expansion underlying the deterministic perfect shuffle. In this setting, the binary skeleton (Figure 3) corresponds to the doubling map $x \mapsto 2x$ (mod 1) studied by Lalley [13], with randomness superimposed to produce a probabilistic analogue of the deterministic dynamics. It also aligns with structures observed in random walks on groups and Markov chains on permutations [14, 15].

Concluding outlook

In addition to the two applications discussed above, the recursive decomposition and closed-form p.g.f.s developed here are not inherently limited to the GSR riffle shuffle. They may also extend to other shuffling schemes or stochastic processes on permutations, such as biased riffle shuffles and top-in-at-random procedures, whenever the one-shuffle distribution admits a tractable decomposition. A natural future direction is to identify the class of permutation Markov chains whose transition kernels possess similar mixture representations, thereby extending the reach of this framework beyond shuffling to broader combinatorial and probabilistic systems.

Author contributions

T. Krityakierne and T. Thanatipanonda: Conceptualization, methodology, investigation, formal analysis, software, validation, visualization, and writing—original draft, review, and editing. All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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