Time for the New Ansatz (?)

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Introduction

Mathematics is the art finding pattern. We also deal with a lot of sequences. Coming up with a pattern of sequence is an essential step.

C-finite Sequence

Definition

The (linear) recurrence relation with only constant coefficients (aka C-finite ansatz, [1, 5]) i.e. the sequence $\{a(n)\}_{n=0}^{\infty}$ where there are constants $c_0, c_1, \ldots, c_{k-2}, c_{k-1}$ such that

$$c_0 a(n) + c_1 a(n+1) + \cdots + c_{k-1} a(n+k-1) + a(n+k) = 0, \quad \text{for all } n \geq 0,$$

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Examples:

1. $F(n) = F(n-1) + F(n-2)$, given that $F(0) = 0, F(1) = 1$.
2. Sequence of polynomials: i.e. $n^2 - 2n$ satisfies $a(n) - 3a(n-1) + 3a(n-2) - a(n-3) = 0, \;\; a(0) = 0, a(1) = -1$.
3. (two dimensional recurrence) $B(n,k) = B(n-1,k) + B(n-1,k-1), \;\; 1 \leq k \leq n-1$ where $B(0,0) = 1, \;\; B(n,0) = B(0,n) = 0, n > 0$.

Holonomic Sequence

Definition

The (linear) recurrence relation with only polynomial coefficients (aka holonomic ansatz, [1, 4]) i.e. the sequence $\{a(n)\}_{n=0}^{\infty}$ where there are polynomials $p_0(n), p_1(n), \ldots, p_{k-1}(n), p_k(n)$, $(p_k(n) \neq 0)$, such that

$$p_0(n)a(n) + p_1(n)a(n+1) + \cdots + p_{k-1}(n)a(n+k-1) + p_k(n)a(n+k) = 0,$$

for all $n \geq 0$.

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for all $n \geq 0$.

Examples:

1. $a(n) = n \cdot a(n-1)$, given that $a(0) = 1$.

Holonomic Sequence (continued)

3. $a_{i,j} = \sum_{s=1}^{j} s^i, \qquad i \geq 0, \; j \geq 1.$

Here are the first couple terms:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ 1 & 3 & 6 & 10 & 15 & 21 & \dots \\ 1 & 5 & 14 & 30 & 55 & 91 & \dots \\ \vdots & & & & & & \end{bmatrix}$$

This sequence is a holonomic sequence:

$$(j+1)(a_{i,j+1} - a_{i,j}) = a_{i+1,j+1} - a_{i+1,j}.$$

Nonlinear Recurrence

Definition

The sequence $\{a(n)\}_{n=0}^{\infty}$ where there are a polynomial with $r+1$ variables such that

$$P(a(n), \ldots, a(n+r)) = 0 \text{ for all } n \geq 1.$$

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Examples: Somos-4

$$a(n) \cdot a(n-4) - a(n-1) \cdot a(n-3) - a(n-2)^2 = 0, \quad n \geq 1$$

where $a(1) = a(2) = a(3) = a(4) = 1$.

New Proposal 1:

Another example:

$$0 = a(n)(a(n+1) \cdot a(n+3) - a(n+2)^2) - a(n+2) \cdot a(n+1)^2, \quad \text{for all } n \geq 0.$$

where $a(0) = 1, a(1) = 1$ and $a(2) = 2$.

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where $a(0) = 1, a(1) = 1$ and $a(2) = 2$.

Here are some of the first terms of the sequence:

$$1, 1, 2, 6, 30, 240, 3120, 65520, 2227680, 122522400, \ldots$$

This sequence is growing too fast to be C-finite or holonomic, but still simple enough for human to detect the pattern. This strongly suggests us to create the new ansatz for this type of sequences.

New Proposal 1: (continued)

Definition (The new ansatz)

A linear recurrence where the coefficients are from holonomic sequences.

$$B_n a_n = C_n a_{n-1} + D_n a_{n-2} + \cdots + Z_n a_{n-k}$$

where each of the sequences $B_n, C_n, D_n, \ldots, Z_n$ are holonomic.

Examples:

1. $a_n = a_{n-1} + 2^n a_{n-2}, \qquad a_0 = a_1 = 1.$
2. $a_n = F_n a_{n-1} + F_{n-1} a_{n-2}, \qquad a_0 = a_1 = 1.$
   (A089126 in Sloane.)

New Proposal 1: (continued)

3. Consider the sequence $a_n := \sum_{i=1}^{n-1} F_i a_i, \qquad a_1 = 1.$

   Some of the first terms are

$$1, 1, 2, 6, 24, 144, 1296, 18144, 399168, 13970880.$$

New Proposal 1: (continued)

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   Some of the first terms are

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   It is not too hard to show that this sequence fits into the new ansatz as follows:

   $$a_n = C_n \cdot a_{n-1}.$$

   where $C_n := 2C_{n-1} - C_{n-3}$ with $C_2 = 3, C_3 = 3$ and $C_4 = 4$.

New Proposal 2:

My second Proposal comes from the example on Schmidt's numbers, which I did during my visit to RISC in 2010, [3].
The problem is as following:

For any integer $r \geq 1$, the sequence of numbers $\{c_k^{(r)}\}_{k \geq 0}$ is defined implicitly by

$$\sum_k \binom{n}{k}^r \binom{n+k}{k}^r = \sum_k \binom{n}{k} \binom{n+k}{k} c_k^{(r)}, \quad n = 0, 1, 2, \ldots$$

In 1992, Asmus Schmidt [2] conjectured that all $c_k^{(r)}$ are integers.

New Proposal 2: (continued)

This fact can be shown by concentrate on each of the term on the left hand side separately.

For $k \geq 0$ and $r \geq 1$, define $a_{k,j}^{(r)}$ as following:

$$\binom{n}{k}^r \binom{n+k}{k}^r = \sum_j a_{k,j}^{(r)} \binom{n}{j} \binom{n+j}{j}.$$

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It is not clear at all that this multi-dimensional sequence $a_{k,j}^{(r)}$ are integers until we discover the non-holonomic recurrence relation of $a_{k,j}^{(r)}$:
$a_{k,k}^{(1)} = 1$, $a_{k,j}^{(1)} = 0$ $(j \neq k)$ and

$$a_{k,j}^{(r+1)} = \sum_i \binom{k+i}{i} \binom{k}{j-i} \binom{j}{k} a_{k,i}^{(r)}.$$

New Proposal 2: Summation Ansatz

The pattern of $a(k, j, r)$ could be found from the ansatz:

$$a(k, j, r + 1) = \sum_i s(k, j, i) a(k, i, r),$$

which resulting in

$$s(k, j, i) = \binom{k + i}{i} \binom{k}{j - i} \binom{j}{k}.$$

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In some simpler situations, the ansatz could have been made as following:

$$a(j,n+1) = \sum_{i} s(j,i)a(i,n),$$

or

$$a(j,n+1) = \sum_{t}\sum_{i} s(j,i,t)a(i,t).$$

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