Ansatz in a Nutshell
A comprehensive step-by-step guide to polynomial, C-finite, holonomic, and C^2-finite sequences

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Abstract. Given a sequence 1, 1, 5, 23, 135, 925, 7285, 64755, 641075, 6993545, 83339745,..., how can we guess a formula for it? This article will quickly walk you through the concept of *ansatz* for classes of polynomial, C-finite, holonomic, and the most recent addition C^2-finite sequences. For each of these classes, we discuss in detail various aspects of the guess and check, generating functions, closure properties, and closed-form solutions. Every theorem is presented with an accessible proof, followed by several examples intended to motivate the development of the theories. Each example is accompanied by a Maple program with the purpose of demonstrating use of the program in solving problems in this area. While this work aims to give a comprehensive review of existing ansatzes, we also systematically fill a research gap in the literature by providing theoretical and numerical results for the C^2-finite sequences.

Keywords: ansatz, C-finite, holonomic, C^2-finite, recurrence relation, generating function, closed-form solution, differential equation

1 Getting started

When we come across a word or a phrase we have never seen before, we look it up in a dictionary. Likewise, whenever we encounter a sequence for which we do not know a formula, we could look it up in the Sloane's OEIS, an online dictionary for number sequences [12]. However, as it is not possible that the OEIS database consists of everyone of them, wouldn't it be great if we could find a formula by ourselves, regardless of whether or not our sequence is there? And this is precisely the central theme of this work.

Theme: Given a sequence $a_n, \quad n = 0, 1, 2, \ldots$, find a (homogeneous) linear recurrence relation of the form:

$$c_r(n)a_{n+r} + c_{r-1}(n)a_{n+r-1} + \cdots + c_0(n)a_n = 0,$$

where $c_i(n)$ could be a constant, a polynomial in n, or even a sequence defined by a linear recurrence (in n) itself.

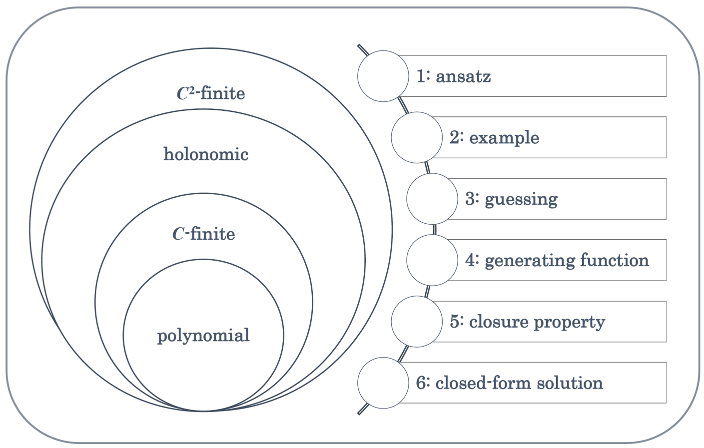

Fig. 1. Left: the classes of sequences in this study; Right: the structure of the paper

This paper is intended to provide a comprehensive background on the concept of ansatz for classes of polynomial, C-finite, holonomic, and the recently developed C^2-finite sequences. The inclusion relation of the four classes is depicted in Figure 1. While there are other classes of sequences (e.g. algebraic sequences, hypergeometric sequences), we focus exclusively on these classes so that we can provide an in-depth review of the subject from both the theoretical and practical aspects. In addition, the reader may notice beauty in the embedding nature of these sequences, i.e. polynomial sequences as coefficients of holonomic recurrences, and C-finite sequences as coefficients of C^2-finite recurrences. The right panel of Figure 1 gives a quick overview of the schematic structure of the paper.

Consequently, the aim of this paper is twofold: firstly, to present a step-by-step guide to ansatzes, and secondly to extend the knowledge of existing ansatzes to the class of C^2-finite sequences in a systematic manner. We have tried to keep the paper as self-contained yet easy-to-follow as possible. Every theorem is presented with a proof, where much effort has been made to make the proof accessible and transparent. Every theorem is illustrated with examples or applications that stimulated the development of the concept. While most of these examples are solved by hand, Maple has been used in several intermediate calculations and simplification of algebraic expressions. At the end of each example, we demonstrate the command required to enter the data of the problem into our Maple program.

While there are several available software tools that provide excellent computing facilities for ansatzes, e.g. Maple package `gfun` [13] and Mathematica package `GeneratingFunctions` [11] which provide commands for dealing with holonomic sequences, our motivation to program everything from scratch is to demonstrate to the readers that it is straightforward enough to follow the steps in the proofs of theorems and write a program to construct problem solutions. In fact, our Maple program followed these steps precisely, and in this way, the provided code serves as a user-friendly guide to interactive programming exercises throughout the paper. The interested reader is invited to study the codes accompanying this paper, provided at https://thotsaporn.com/Ansatz.html or implement the program in their favorite programming environment. The output of each Maple program can also be found on the website.

A walk-through example: an ansatz is a guess

Before delving into the next topic, let us first understand what we mean by "guessing". Suppose we encounter a sequence $a_n = \sum_{i=0}^{n} i^2$ while solving a problem, and we wish to find a closed-form expression $f(n)$ for the nth term. Here, we are going to make a few assumptions. First, we will assume that a_n is a polynomial sequence, i.e.

$$a_n = c_k n^k + c_{k-1} n^{k-1} + \cdots + c_0,$$

for some (unknown) degree k and coefficients $c_0, c_1, \ldots, c_k$.

Since the formula $a_n = \sum_{i=0}^{n} i^2$ is given, we can generate as many terms in the sequence as we want, as all we need to do is plug the values $n = 0, 1, 2, 3, \ldots$ into a_n, which gives $(a_n)_{n=0}^{\infty} = 0, 1, 5, 14, 30, 55, 91, 140, \ldots$.

These values will play the role of dataset for polynomial curve "fitting" and "checking". Since the exact degree k of the polynomial is not known, we have to assume it as part of the guess. As we try to keep the degree of the polynomial model as low as possible, let us start with $k = 2$. In this case, we will need (at least) three data points to fit the model. Solving for the coefficients c_0, c_1, c_2, the system of equations

$$0 = c_2 \cdot 0^2 + c_1 \cdot 0^1 + c_0 \tag{1}$$

$$1 = c_2 \cdot 1^2 + c_1 \cdot 1^1 + c_0 \tag{2}$$

$$5 = c_2 \cdot 2^2 + c_1 \cdot 2^1 + c_0, \tag{3}$$

yields $c_0 = 0, c_1 = -1/2, c_2 = 3/2$, i.e. $f(n) = \frac{3}{2}n^2 - \frac{1}{2}n$. Unfortunately, $f(3) = 12 \neq 14 = a_3$, so the second-order polynomial interpolation result is not satisfactory. By keeping on increasing the degree until the fitted polynomial model can predict the values of the next term(s) correctly, we found the sought-after formula $f(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ which makes $f(n) = a_n$ for all $n \geq 0$. Of course, this last claim has to be verified, for example, by induction.

What we just did here is we first make an educated guess (ansatz), then solve for the coefficients, and later we verify the expression a_n for all n. This is precisely the guess and check method. Note however that if the formula $a_n = \sum_{i=0}^{n} i^2$ is not provided, but we are given only the values of the first five terms: 0, 1, 5, 14, 30, we can still proceed in the same way as before, using the first three terms $0, 1, 5$ to obtain the third degree polynomial. However, this time we only have two terms $14, 30$ left for verification. And this is the best we can do given the limited data available.

The outline of the rest of this paper is as follows. In the next section, we give a comprehensive review of the existing ansatz along with important theoretical properties. Section 3 presents results related to C^2-finite sequences where we formally give a definition of C^2-finite, and establish several results concerning the generating function and the closure properties. We also list a few interesting unsolved problems in that section. Through our presentation, we hope that this paper will provide a unifying framework for understanding the various aspects of ansatz in the classes of polynomial, C-finite, holonomic, and C^2-finite sequences.

2 Comprehensive review of existing ansatz

This section concentrates on a comprehensive review of ansatzes in the classes of polynomials (as a sequence), C-finite, and holonomic sequences. For a list of excellent resources, see for example, [9,18] for C-finite, [7] for holonomic, as well as [2,8] and the bibliography. Although this list of references is by no means exhaustive, we acknowledge the contributions of all pioneer researchers in the field.

2.1 Polynomial as a sequence

1. **Ansatz:** $a_n = c_k n^k + c_{k-1} n^{k-1} + \cdots + c_0$.
2. **Example:** Let $a_n = \displaystyle\sum_{i=1}^{n} i^2$. Here, $a_n = \dfrac{n^3}{3} + \dfrac{n^2}{2} + \dfrac{n}{6} = \dfrac{n(n+1)(2n+1)}{6}$.
3. **Guessing:** The goal is to find the polynomial of lowest possible degree that passes through the dataset.

INPUT: the assumed degree k of polynomial and the dataset sequence a_n (of length more than $k+1$).
We solve the system of linear equations for $c_0, c_1, \ldots, c_k$:

$$\begin{bmatrix} 1 & 0 & & 0^k \\ 1 & 1 & \cdots & 1^k \\ 1 & 2 & \cdots & 2^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & k & & k^k \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{bmatrix}.$$

The existence of the inverse of the Vandermonde matrix implies the existence and uniqueness of solutions. Once all these c_i's are obtained, we have to verify that our conjectured polynomial is valid for the rest of the available terms $a_n, n > k$ in the sequence; otherwise there is no solution. In this case, we increment the degree by 1 and refit the polynomial.

Maple Program

```
> A:= [seq(add(i^2,i=0..n),n=0..20)];
> GuessPol(A,0,n);
```

4. **Linear recurrence:**
The following proposition provides a linear recurrence relation for a polynomial sequence. For convenience, we define the *left shift operator* on a_n by $Na_n = a_{n+1}$.

Proposition 1. *a_n is a polynomial in n of degree at most k if and only if $(N-1)^{k+1}a_n = 0$.*

Proof. Assume a_n is a polynomial in n of degree k. Observe that $(N-1)a_n = a_{n+1} - a_n$ is a polynomial whose degree is reduced by (at least) 1. By applying the operator $(N-1)$ repeatedly $k+1$ times, we obtain the homogeneous linear recurrence relation of a_n, i.e.

$$(N-1)^{k+1}a_n = 0.$$

This completes the proof of the forward direction. For the reverse direction, suppose the condition $(N-1)^{k+1}a_n = 0$ holds. Consider

$$a_n = N^n a_0 = [1 + (N-1)]^n a_0 = \sum_{i=0}^{n} \binom{n}{i} (N-1)^i a_0.$$

It follows from the assumption that

$$a_n = \sum_{i=0}^{k} \binom{n}{i} (N-1)^i a_0. \tag{4}$$

Hence, a_n is a polynomial (in n) of degree at most k. □

We obtain the following corollary immediately from (4). Note that this has also been discussed in Proposition 1.4.2 of [15].

Corollary 2. *A polynomial a_n of degree at most k can be written in expanded form in terms of the binomial basis as*

$$a_n = \sum_{i=0}^{k} \binom{n}{i} (N-1)^i a_0.$$

Example: Let $a_n = \dfrac{n(n+1)(2n+1)}{6}$. Then, a_n can be written as $2\binom{n}{3} + 3\binom{n}{2} + \binom{n}{1}$.

An important consequence of this observation is the following.

Proposition 3. *For a non-negative integer k, $\sum_{i=0}^{n} i^k$ is a polynomial of degree at most $k+1$.*

Proof. This is easy to see. Let $a_n = \sum_{i=0}^{n} i^k$. Then, $(N-1)a_n = a_{n+1} - a_n = (n+1)^k$, a polynomial of degree k. Hence, from Proposition 1, $(N-1)^{k+2}a_n = 0$ and, by reverse observation of Proposition 1, a_n is a polynomial of degree at most $k+1$. □

Remark. Proposition 3 is very useful as knowing such a bound on the polynomial degree allows us to make guesses rigorous. In particular, finding a polynomial equation for $a_n = \sum_{i=0}^{n} i^k$, for some fixed k, amounts to fitting a polynomial of degree $k+1$ to a set of data points a_n, $n = 0, 1, 2, \ldots, k+1$.

5. Generating function:

Every sequence corresponds to a generating function that comes in handy when determining a formula of the sequence, as we shall see later. Let us note that their generating function considered here is a *formal* power series in the sense that it is regarded as an algebraic object, thereby ignoring the issue of convergence. The next proposition establishes a connection between polynomial sequences and the generating functions.

Proposition 4. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where a_n is a polynomial in n of degree k. Then,*

$$f(x) = \frac{P(x)}{(1-x)^{k+1}}, \tag{5}$$

for some polynomial $P(x)$ of degree at most k.

Proof. Assume a_n is a polynomial in n of degree k. Then,

$$\begin{aligned}(1-x)^{k+1}f(x) &= \sum_{n=0}^{\infty} a_n x^n (1-x)^{k+1} \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i a_n x^{n+i} \\ &= \sum_{i=0}^{k+1}\sum_{n=i}^{\infty}\binom{k+1}{i}(-1)^i a_{n-i} x^{n} \\ &= \sum_{n=k+1}^{\infty}\sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i a_{n-i} x^{n} + \sum_{n=0}^{k}\sum_{i=0}^{n}\binom{k+1}{i}(-1)^i a_{n-i} x^{n}.\end{aligned}$$

The first summation is essentially zero as, for each $n \geq k+1$,

$$\sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i a_{n-i} = (N-1)^{k+1} a_{n-k-1} = 0,$$

by Proposition 1. Hence,

$$(1-x)^{k+1}f(x) = P(x),$$

where

$$P(x) = \sum_{n=0}^{k}\sum_{i=0}^{n}\binom{k+1}{i}(-1)^i a_{n-i} x^{n}, \qquad (6)$$

a polynomial of degree at most k. □

Example: The generating function $f(x)$ for $a_n = \displaystyle\sum_{i=1}^{n} i^2$ is $\dfrac{x^2+x}{(1-x)^4}$.

Maple Program

```
> GenPol(n*(n+1)*(2*n+1)/6,n,x);
```

The next proposition states the converse of Proposition 4.

Proposition 5. *Assume that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *satisfies*

$$f(x) = \frac{P(x)}{(1-x)^{k+1}},$$

for some polynomial $P(x)$ *of degree* k*. Then,* a_n *is a polynomial sequence of degree at most* k*.*

Proof. Since

$$(1-x)^{k+1}f(x) = \sum_{n=k+1}^{\infty}\sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i a_{n-i}x^n + \sum_{n=0}^{k}\sum_{i=0}^{n}\binom{k+1}{i}(-1)^i a_{n-i}x^n$$

is a polynomial of degree k, the first summation must be zero implying $(N-1)^{k+1}a_{n-k-1} = 0$ for all $n \geq k+1$. Hence, from Proposition 1, we can conclude that a_n is a polynomial of degree at most k. □

6. **Closure properties:** The closure properties presented below and (in later sections) hold in any field with either positive or zero characteristic. We thus state the most general results. Of course, the term "at most" in properties (ii)-(v) can be dropped if one works over a field of characteristic zero as the degree becomes exact in this case.

Theorem 6. *Assume that a_n and b_n are polynomial sequences of degree k and l, respectively. The following sequences are also polynomial sequences.*
(i) addition: $(a_n + b_n)_{n=0}^{\infty}$, degree at most $\max(k,l)$,
(ii) term-wise multiplication: $(a_n \cdot b_n)_{n=0}^{\infty}$, degree at most $k+l$,
(iii) Cauchy product: $(\sum_{i=0}^{n} a_i \cdot b_{n-i})_{n=0}^{\infty}$, degree at most $k+l+1$,
(iv) partial sum: $(\sum_{i=0}^{n} a_i)_{n=0}^{\infty}$, degree at most $k+1$,
(v) linear subsequence: $(a_{mn})_{n=0}^{\infty}$, where m is a positive integer, degree at most k.

Proof. The proofs of claims *(i)*,*(ii)*, and *(v)* are rather straightforward. Claims *(iii)* and *(iv)* follow from Proposition 3 that $\sum_{i=0}^{n} i^k$, where k is a non-negative integer, is a polynomial in n of degree at most $k+1$. □

2.2 C-finite

1. **Ansatz:** a_n is defined by a homogeneous linear recurrence with constant coefficients:

$$a_{n+r} + c_{r-1}a_{n+r-1} + c_{r-2}a_{n+r-2} + \cdots + c_0 a_n = 0,$$

along with the initial values $a_0, a_1, \ldots, a_{r-1}$. We call a_n a *C-finite sequence* of order r.

Remark. From Proposition 1, a polynomial sequence is a special case of C-finite sequences.

2. **Example:** Let $a_n = \left\lfloor \left(\frac{n}{2}\right)^2 \right\rfloor$. Here, a_n satisfies the linear recurrence relation of order 4:

$$a_{n+4} - 2a_{n+3} + 2a_{n+1} - a_n = 0,$$

with initial values $a_0 = 0, a_1 = 0, a_2 = 1$ and $a_3 = 2$. In terms of the left shift operator,

$$0 = (N^4 - 2N^3 + 2N - 1) \cdot a_n = (N+1)(N-1)^3 \cdot a_n.$$

3. Guessing:

INPUT: the order r of linear recurrence and sequence a_n (of length more than $2r$). We solve the system of linear equations for $c_0, c_1, \ldots, c_{r-1}$:

$$\begin{bmatrix} a_0 & a_1 & & a_{r-1} & a_r \\ a_1 & a_2 & & a_r & a_{r+1} \\ a_2 & a_3 & \ddots & a_{r+1} & a_{r+2} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{r-1} & a_r & & a_{2r-2} & a_{2r-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{r-1} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

This matrix equation can be solved quickly through, say, the reduced row echelon form method. Once all these c_i's are obtained, we check that the rest of the $a_n, n > 2r-1$ satisfy the conjectured recurrence.

Maple Program

```
> A:= [seq(floor((n/2)^2), n=0..30)];
> GuessC(A,N);
```

4. Generating function:

Let us denote by $T(N)$ an *annihilator* of a_n, that is, $T(N) \cdot a_n = 0$. In line with Proposition 4 in the previous section, the following proposition establishes a relationship between C-finite sequences and generating functions.

Proposition 7. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where a_n is a C-finite sequence of order r with annihilator $T(N)$, a polynomial in N of degree r. Then,*

$$f(x) = \frac{P(x)}{x^r T(1/x)}, \tag{7}$$

for some polynomial $P(x)$ of degree at most $r-1$.

Proof. Assume that a_n is a C-finite sequence with annihilator $T(N)$. Suppose further that $T(N) = c_r N^r + c_{r-1}N^{r-1} + \cdots + c_1 N + c_0$, where $c_r = 1$ and $c_{r-1}, \ldots, c_1, c_0$ are some constants. Then,

$$\begin{aligned} x^r T(1/x) f(x) &= \sum_{n=0}^{\infty} a_n x^n (c_0 x^r + c_1 x^{r-1} + \cdots + c_r) \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{r} c_i a_n x^{n+r-i} = \sum_{i=0}^{r} \sum_{n=r-i}^{\infty} c_i a_{n-r+i} x^n \\ &= \sum_{n=r}^{\infty} \sum_{i=0}^{r} c_i a_{n-r+i} x^n + \sum_{n=0}^{r-1} \sum_{i=r-n}^{r} c_i a_{n-r+i} x^n. \end{aligned}$$

The first sum is zero as, for each $n \geq r$,

$$\sum_{i=0}^{r} c_i a_{n-r+i} = \sum_{i=0}^{r} c_i N^i a_{n-r} = T(N) \cdot a_{n-r} = 0,$$

by assumption of $T(N)$. Hence,

$$x^r T(1/x) f(x) = P(x),$$

where $P(x) = \sum_{n=0}^{r-1} \sum_{i=r-n}^{r} c_i a_{n-r+i} x^n$, a polynomial of degree at most $r-1$. □

Example: The generating function $f(x)$ for $a_n = \left\lfloor \left(\frac{n}{2}\right)^2 \right\rfloor$ is $\dfrac{x^2}{(1+x)(1-x)^3}$.

Maple Program

```
> GenC(N^4-2*N^3+2*N-1,[0,0,1,2],N,x);
```

The next proposition gives the converse of Proposition 7. This proposition is very useful as it allows us to prove closure properties through generating functions which turned out to simplify our proof tremendously. In particular, the proposition implies that an upper bound for the order of a C-finite recurrence can be determined by looking at the degree of the polynomial appearing in the denominator of (7).

Proposition 8. *Assume that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *satisfies*

$$f(x) = \frac{P(x)}{x^r T(1/x)},$$

for some polynomial $P(x)$ *of degree* $r-1$*, and* $T(N)$ *is a polynomial of degree* r*. Then,* a_n *is a C-finite sequence of order at most* r*.*

Proof. Since

$$x^r T(1/x) f(x) = \sum_{n=r}^{\infty} \sum_{i=0}^{r} c_i a_{n-r+i} x^n + \sum_{n=0}^{r-1} \sum_{i=r-n}^{r} c_i a_{n-r+i} x^n$$

is a polynomial of degree $r-1$, the first summation must be zero implying $T(N) \cdot a_{n-r} = 0$ for all $n \geq r$. Thus, $T(N)$ is an annihilator of a_n, and so a_n is a C-finite sequence of order at most r. □

5. Closure properties:

It will become evident later that knowing the operations under which the class of C-finite sequences is closed allows one to guess rigorously a formula of the resulting sequence. We first state and prove the following closure properties.

Theorem 9. *Assume that a_n and b_n are C-finite sequences of order r and s, respectively. The following sequences are also C-finite, with the specified upper bound on the order.*

(i) addition: $(a_n + b_n)_{n=0}^{\infty}$, order at most $r + s$,
(ii) term-wise multiplication: $(a_n \cdot b_n)_{n=0}^{\infty}$, order at most rs,
(iii) Cauchy product: $(\sum_{i=0}^{n} a_i \cdot b_{n-i})_{n=0}^{\infty}$, order at most $r + s$,
(iv) partial sum: $(\sum_{i=0}^{n} a_i)_{n=0}^{\infty}$, order at most $r + 1$,
(v) linear subsequence: $(a_{mn})_{n=0}^{\infty}$, where m is a positive integer, order at most r.

Proof. The proofs for the closure properties are based on two different approaches, i.e. the generating function approach for proving *(i)*, *(iii)* and *(iv)*, and the solution subspace approach for *(ii)* and *(v)*.

Generating function approach
To prove the closure properties of addition, Cauchy product and partial sum, let $A(x)$ and $B(x)$ be the generating functions of a_n and b_n, respectively. Then, the generating functions $C(x)$ of $c_n = a_n + b_n, \sum_{i=0}^{n} a_i \cdot b_{n-i}$ and $\sum_{i=0}^{n} a_i$ are $A(x)+B(x), A(x)B(x)$ and $A(x) \cdot \dfrac{1}{1-x}$, respectively. By Proposition 8, c_n is a C-finite sequence whose order can be determined by looking at the degree of the polynomial appearing in the denominator of $C(x)$ in each case.

□

We now give concrete examples of how the generating function approach can be used to verify the closure properties of *(i)*, *(iii)* and *(iv)*.

Example: Let $a_n = \left\lfloor \left(\dfrac{n}{2}\right)^2 \right\rfloor$ and b_n be the Fibonacci numbers.
We recall that a_n satisfies the linear recurrence relation

$$a_{n+4} - 2a_{n+3} + 2a_{n+1} - a_n = 0,$$

with $a_0 = 0, a_1 = 0, a_2 = 1$ and $a_3 = 2$. In terms of the shift operator,

$$(N+1)(N-1)^3 \cdot a_n = 0.$$

Also, b_n satisfies the linear recurrence relation

$$b_{n+2} - b_{n+1} - b_n = 0,$$

with $b_0 = 0$ and $b_1 = 1$. In terms of the shift operator,

$$(N^2 - N - 1) \cdot b_n = 0.$$

The generating functions of a_n and b_n are $A(x) = \dfrac{x^2}{(1+x)(1-x)^3}$, and $B(x) = \dfrac{x}{1-x-x^2}$, respectively.

- addition: $c_n = a_n + b_n$. Then,

$$C(x) = A(x) + B(x) = \frac{-x(x^4 - x^3 + x^2 + x - 1)}{(1+x)(1-x)^3(1-x-x^2)}.$$

That is, c_n satisfies the linear recurrence of order 6:

$$(N+1)(N-1)^3(N^2 - N - 1) \cdot c_n = 0.$$

- Cauchy product: $c_n = \sum_{i=0}^{n} a_i \cdot b_{n-i}$. Then,

$$C(x) = A(x)B(x) = \frac{x^3}{(1+x)(1-x)^3(1-x-x^2)}.$$

That is, c_n satisfies the linear recurrence of order 6:

$$(N+1)(N-1)^3(N^2 - N - 1) \cdot c_n = 0.$$

- partial sum: $c_n = \sum_{i=0}^{n} a_i$ then

$$C(x) = A(x) \cdot \frac{1}{1-x} = \frac{x^2}{(1+x)(1-x)^4}.$$

That is, c_n satisfies the linear recurrence of order 5:

$$(N+1)(N-1)^4 \cdot c_n = 0.$$

Maple Program

```
> A := x^2/(1+x)/(1-x)^3:
> B := x/(1-x-x^2):
> CAddition(A,B,x);
> CCauchy(A,B,x);
> CParSum(A,x);
```

Solution space approach

Before embarking on a proof for closure properties of linear subsequence and term-wise multiplication, we continue with the example from the previous section, intended to highlight the key steps of the solution space approach. A formal proof will be deferred to the end of this section.

- linear subsequence: $c_n = a_{2n}$. First, we apply the linear recurrence relation of a_{2n}, i.e. $a_{2n+4} = 2a_{2n+3} - 2a_{2n+1} + a_{2n}$ repeatedly to yield

$$\begin{bmatrix} c_n \\ c_{n+1} \\ c_{n+2} \\ c_{n+3} \\ c_{n+4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 2 \\ 4 & -6 & -3 & 6 \\ 9 & -12 & -8 & 12 \end{bmatrix} \cdot \begin{bmatrix} a_{2n} \\ a_{2n+1} \\ a_{2n+2} \\ a_{2n+3} \end{bmatrix}.$$

A solution $P = [-1, 3, -3, 1, 0]$ is in the left null space of matrix M (in the middle), i.e. $P \cdot M = (0, 0, 0, 0)$. Hence, the linear recurrence of c_n is

$$-c_n + 3c_{n+1} - 3c_{n+2} + c_{n+3} = 0$$

or in terms of the shift operator

$$(N-1)^3 \cdot c_n = 0.$$

Maple Program

```
> CSubSeq(2,N^4-2*N^3+2*N-1,N);
```

Note that the above program returns $(N-1)^3(Nd_4-1)\cdot c_n = 0$, where d_4 is a free variable. We can assign zero to d_4 and obtain the desired third order recurrence relation.

- term-wise multiplication: $c_n = a_n \cdot b_n$. We apply the linear recurrence relation of a_n and b_n, i.e. $a_{n+4} = 2a_{n+3} - 2a_{n+1} + a_n$ and $b_{n+2} = b_{n+1} + b_n$ repeatedly to yield the system of relations

$$\begin{aligned} a_nb_n &= 1a_nb_n + 0a_nb_{n+1} + 0a_{n+1}b_n + 0a_{n+1}b_{n+1} + \cdots + 0a_{n+3}b_{n+1}, \\ a_{n+1}b_{n+1} &= 0a_nb_n + 0a_nb_{n+1} + 0a_{n+1}b_n + 1a_{n+1}b_{n+1} + \cdots + 0a_{n+3}b_{n+1}, \\ &\vdots \\ a_{n+8}b_{n+8} &= 117a_nb_n + 189a_nb_{n+1} - 156a_{n+1}b_n - 252a_{n+1}b_{n+1} + \ldots \\ &\quad + 252a_{n+3}b_{n+1}. \end{aligned}$$

We put this system of equations in the matrix form as

$$\begin{bmatrix} c_n \\ c_{n+1} \\ c_{n+2} \\ c_{n+3} \\ c_{n+4} \\ c_{n+5} \\ c_{n+6} \\ c_{n+7} \\ c_{n+8} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 2 & 3 & -4 & -6 & 0 & 0 & 4 & 6 \\ 6 & 10 & -9 & -15 & -6 & -10 & 12 & 20 \\ 20 & 32 & -30 & -48 & -15 & -24 & 30 & 48 \\ 48 & 78 & -64 & -104 & -48 & -78 & 72 & 117 \\ 117 & 189 & -156 & -252 & -104 & -168 & 156 & 252 \end{bmatrix} \cdot \begin{bmatrix} a_nb_n \\ a_nb_{n+1} \\ a_{n+1}b_n \\ a_{n+1}b_{n+1} \\ a_{n+2}b_n \\ a_{n+2}b_{n+1} \\ a_{n+3}b_n \\ a_{n+3}b_{n+1} \end{bmatrix}.$$

A non-trivial solution $[1, 2, -4, -8, 5, 8, -4, -2, 1]$ is in the left null space of matrix M (in the middle). Hence, the linear recurrence of c_n is

$$c_n + 2c_{n+1} - 4c_{n+2} - 8c_{n+3} + 5c_{n+4} + 8c_{n+5} - 4c_{n+6} - 2c_{n+7} + c_{n+8} = 0$$

or in terms of the shift operator

$$(N^2 + N - 1)(N^2 - N - 1)^3 \cdot c_n = 0.$$

Maple Program

```
> CTermWise(N^4-2*N^3+2*N-1,N^2-N-1,N);
```

We shall now give a formal proof of the closure properties of linear subsequence and term-wise multiplication in the spirit of the last two examples.

Proof. For the case of the linear subsequence with a fixed positive integer m, let $c_n = a_{mn}$. Then $c_n, c_{n+1}, \dots, c_{n+r}$ can be put in the system of linear equations as

$$\begin{bmatrix} c_n \\ c_{n+1} \\ c_{n+2} \\ \cdots \\ c_{n+r} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ & & & \dots \\ M_{r-1}^{(1)} & M_{r-1}^{(2)} & M_{r-1}^{(3)} & \dots \\ M_r^{(1)} & M_r^{(2)} & M_r^{(3)} & \dots \end{bmatrix} \cdot \begin{bmatrix} a_{mn} \\ a_{mn+1} \\ a_{mn+2} \\ \cdots \\ a_{mn+r-1} \end{bmatrix}.$$

The constant matrix M (in the middle) has $r+1$ rows and r columns, which guarantees a non-trivial null space, i.e. there exists a solution $P \neq 0$ such that $P \cdot M = [0\,0 \cdots 0]$. This solution P provides a C-finite recurrence relation to $c_n, c_{n+1}, \dots, c_{n+r}$.
As for the term-wise multiplication, let $c_n = a_n \cdot b_n$. Then, $c_n, c_{n+1}, \dots, c_{n+rs}$ can be put in the system of linear equations as

$$\begin{bmatrix} c_n \\ c_{n+1} \\ c_{n+2} \\ \cdots \\ c_{n+rs-1} \\ c_{n+rs} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots\ 0\ 0\ 0 \dots \\ 0 & 0 & 0 & \dots\ 0\ 1\ 0 \dots \\ & & & \dots \\ M_{rs-1}^{(1)} & M_{rs-1}^{(2)} & M_{rs-1}^{(3)} & \dots \\ M_{rs}^{(1)} & M_{rs}^{(2)} & M_{rs}^{(3)} & \dots \end{bmatrix} \cdot \begin{bmatrix} a_n b_n \\ a_n b_{n+1} \\ \cdots \\ a_{n+1} b_n \\ a_{n+1} b_{n+1} \\ \cdots \\ a_{n+r-1} b_{n+s-1} \end{bmatrix}.$$

The constant matrix M (in the middle) has $rs+1$ rows and rs columns, which again guarantees a non-trivial solution P in the null space of M. This solution P gives a C-finite recurrence relation to $c_n, c_{n+1}, \dots, c_{n+rs}$. □

Rigorous proof of identities with the closure properties
The closure properties from Theorem 9 are extremely useful in verifying identities of C-finite sequences. For example, let $c_n = a_n^2$, where $a_n = \left\lfloor \left(\frac{n}{2}\right)^2 \right\rfloor$. Let us say we want to show that c_n satisfies a linear recurrence

$$c_{n+8} = 2c_{n+7} + 2c_{n+6} - 6c_{n+5} + 6c_{n+3} - 2c_{n+2} - 2c_{n+1} + c_n, \quad n \geq 0,$$

i.e. $(N+1)^3(N-1)^5 \cdot c_n = 0$. Knowing that a_n satisfies the linear recurrence of order 4, Theorem 9: term-wise multiplication guarantees that

c_n satisfies the linear recurrence of order at most $4 \cdot 4 = 16$. It then suffices to verify the identity only by checking the numeric values for $n = 0, 1, 2, \ldots, 15$ as this is adequate to determine a C-finite recurrence of order 16.
Let us give another example. Consider the same sequence a_n, but this time we want to verify a non-linear identity

$$a_{n+1} - a_n a_{n+1} + a_n a_{n+2} + a_{n+1}^2 - a_{n+1}a_{n+2} = 0, \quad n \geq 0.$$

We define $d_n = a_{n+1} - a_n a_{n+1} + a_n a_{n+2} + a_{n+1}^2 - a_{n+1}a_{n+2}$ for $n \geq 0$. The closure properties of C-finite sequence ensure that the order of d_n will be at most $4 + 4^2 + 4^2 + 4^2 + 4^2 = 68$. So to prove that $d_n = 0$, for $n \geq 0$, we only check the initial values of d_n for $n = 0, 1, 2, \ldots, 67$.

6. **Closed-form solutions:**
In contrast to polynomial sequences that we started with an ansatz in a closed-form solution (and later derived a linear recurrence relation for it), our ansatz for a C-finite sequence was initially defined as a recurrence relation. A closed-form representation will now be given in the following proposition.

Proposition 10. *Assume that a_n satisfies the linear recurrence with constant coefficients,*

$$0 = T(N) \cdot a_n = [(N - \alpha_1)^{k_1}(N - \alpha_2)^{k_2} \ldots (N - \alpha_m)^{k_m}] \cdot a_n,$$

where α_i, the i-th root of the characteristic polynomial of the recurrence, has multiplicity k_i, $i = 1, 2, \ldots, m$. Then, a closed-form formula of a_n is given by

$$a_n = \sum_{i=1}^{m} (c_{i,0} + c_{i,1}n + \cdots + c_{i,k_i-1}n^{k_i-1})\alpha_i^n, \tag{8}$$

for some constants $c_{i,0}, c_{i,1}, \ldots, c_{i,k_i-1}$, $\quad i = 1, 2, \ldots, m$, which can be determined by the initial conditions.

Proof. We will show that $T(N) \cdot \sum_{i=1}^{m}(c_{i,0} + c_{i,1}n + \cdots + c_{i,k_i-1}n^{k_i-1})\alpha_i^n = 0$. This can be done by showing that each part of the sum is 0, i.e. for each i,

$$(N - \alpha_i)^{k_i} \cdot [(c_{i,0} + c_{i,1}n + \cdots + c_{i,k_i-1}n^{k_i-1})\alpha_i^n] = 0.$$

Notice that (dropping the i subscript for simplicity)

$$(N - \alpha) \cdot [P(n)\alpha^n] = P(n+1)\alpha^{n+1} - P(n)\alpha^{n+1} = Q(n)\alpha^{n+1},$$

where $Q(n)$ has degree less than $P(n)$. Therefore, if we apply $(N - \alpha)$ for k times to $P(n)\alpha^n$ where $P(n)$ is a polynomial of degree $k - 1$, we will get 0, and the result is now immediate. We note that this formula of a_n is the most general form as the number of independent solutions equals the degree of $T(N)$. □

Example: A closed-form formula for $a_n = \left\lfloor \left(\frac{n}{2}\right)^2 \right\rfloor$ is $\frac{n^2}{4} - \frac{1}{8} + \frac{(-1)^n}{8}$.

Maple Program

```
> CloseC(N^4-2*N^3+2*N-1,[0,0,1,2],0,N,n);
```

The next proposition, which is the converse of the previous proposition, ensures that any sequence expressed in the form (8) is also a C-finite sequence. This proposition will be used in a later section to prove some results for C^2-finite sequences.

Proposition 11. *Suppose that*

$$a_n = \sum_{i=1}^{m} (c_{i,0} + c_{i,1}n + \cdots + c_{i,k_i-1}n^{k_i-1})\alpha_i^n,$$

for some constants $c_{i,0}, c_{i,1}, \ldots, c_{i,k_i-1}$, and α_i, $i = 1, 2, \ldots, m$. Then, a_n satisfies a linear recurrence with constant coefficients, i.e. a_n is a C-finite sequence. Moreover, the order of a_n is $k_1 + k_2 + \cdots + k_m$.

Proof. Adopt the same steps followed in the proof of Proposition 10 to get

$$[(N - \alpha_1)^{k_1}(N - \alpha_2)^{k_2} \ldots (N - \alpha_m)^{k_m}] \cdot a_n = 0.$$

This relation together with initial values a_n of length $k_1 + k_2 + \cdots + k_m$ defines a C-finite recurrence relation for a_n. □

2.3 Holonomic

1. **Ansatz:** a_n is defined by a linear recurrence with polynomial coefficients:

$$p_r(n)a_{n+r} + p_{r-1}(n)a_{n+r-1} + \cdots + p_0(n)a_n = 0,$$

where $p_r(n) \neq 0, n = 0, 1, 2, \ldots$, along with the initial values $a_0, a_1, \ldots, a_{r-1}$. We call a_n a *holonomic sequence* of order r and degree k, where k is the highest degree amongst the polynomials $p_r(n), p_{r-1}(n), \ldots, p_0(n)$.
It is important to note that the condition $p_r(n) \neq 0$ for the leading coefficient is necessary for recursively computing the term a_{n+r} in the sequence from its predecessors. In case of violation of the condition, the relation will be valid for $a_n, n > n_0$ where n_0 is the largest positive integer root of the equation $p_r(n) = 0$.

2. **Example 1:** Let $a_n = \frac{1}{n+1}\binom{2n}{n}$, the Catalan numbers. Here, a_n satisfies a holonomic recurrence of order 1 and degree 1:

$$(4n + 2)a_n - (n + 2)a_{n+1} = 0,$$

with initial value $a_0 = 1$. In terms of the shift operator,

$$[(4n+2) - (n+2)N] \cdot a_n = 0.$$

Example 2: Let $a_n = \sum_{i=1}^{n} \frac{1}{i}$, the harmonic numbers. Here, a_n satisfies a holonomic recurrence of order 2 and degree 1:

$$(n+1)a_n - (2n+3)a_{n+1} + (n+2)a_{n+2} = 0,$$

with initial values $a_1 = 1, a_2 = \frac{3}{2}$. In terms of the shift operator,

$$[n+1-(2n+3)N+(n+2)N^2] \cdot a_n = 0.$$

Example 3: Let $a_n = \left\lfloor \left(\frac{n}{2}\right)^2 \right\rfloor$, from the previous section. Here, a_n also satisfies a holonomic recurrence of order 2 and degree 1:

$$(n+2)a_n + 2a_{n+1} - na_{n+2} = 0.$$

This example illustrates the trade-off between lower order and higher degree compared to the C-finite recurrence in the previous section. The interested reader is referred to e.g. [8] for the discussion about trading the order for degree.

3. Guessing:

INPUT: the order r and degree k of linear recurrence and sequence a_n of length more than $\gamma - 1 + r$, where $\gamma := (r+1)(k+1)$. Let us write $p_i(n) = \sum_{j=0}^{k} c_{i,j} n^j$, $C_i = \left[c_{i,0}, c_{i,1}, \ldots, c_{i,k}\right]$ and $P_n = \left[1, n, n^2, \ldots, n^k\right]$.
In matrix notation, the system of linear equations for $c_{i,j}, 0 \leq i \leq r, 0 \leq j \leq k$ takes the following form:

$$\begin{bmatrix} P_0 a_0 & P_0 a_1 & & P_0 a_r \\ P_1 a_1 & P_1 a_2 & & P_1 a_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ P_{\gamma-1} a_{\gamma-1} & P_{\gamma-1} a_{\gamma} & & P_{\gamma-1} a_{\gamma-1+r} \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

A non-trivial solution for the $c_{i,j}$'s can be found quickly using e.g. the reduced row echelon form method. Once we get all these $c_{i,j}$'s, we check with the rest of the $a_n, n > (r+1)(k+1) - 1 + r$, that they satisfy the conjectured recurrence.

Maple Program

```
> A:= [seq(add(1/i,i=1..n),n=1..35)];
> GuessHo(A,2,1,1,n,N);
```

Exercise: The sequence 1, 1, 5, 23, 135, 925, 7285, 64755, 641075, 6993545, 83339745,..., appearing in the abstract, is not contained in the OEIS database. Convince yourself that this sequence has a recurrence relation of the form

$$a_{n+2} = (n+3)a_{n+1} + (n+2)a_n$$

with $a_0 = a_1 = 1$.

Guessing turns into Rigorous Proof
We note that guessing can turn into a rigorous proof, if we happen to know the (upper bounds of the) order and degree of the relation of the holonomic sequence, as illustrated in the following example. The subject of finding bounds of the order and degree will be discussed later in Theorem 15 and the remark on page 27, respectively.
Example: Given that we know a_n is a holonomic sequence of order 2 and degree 3, then a_n must satisfy the recurrence relation

$$(c_{2,0} + c_{2,1}n + c_{2,2}n^2 + c_{2,3}n^3) \cdot a_{n+2} + (c_{1,0} + c_{1,1}n + c_{1,2}n^2 + c_{1,3}n^3) \cdot a_{n+1}$$
$$+ (c_{0,0} + c_{0,1}n + c_{0,2}n^2 + c_{0,3}n^3) \cdot a_n = 0,$$

for some unknown $c_{i,j}, 0 \le i \le 2, 0 \le j \le 3$. To find the holonomic relation, we simply fit this equation to some data a_n, where we need at least a_n, $n = 0, 1, \ldots, 13$ to solve for the 12 unknowns.

4. **Generating function:**

Theorem 12. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where a_n is a holonomic sequence of order r and degree k. Then, $f(x)$ satisfies the (non-homogeneous) linear differential equation with polynomial coefficients*

$$q_0(x)f(x) + q_1(x)f'(x) + \cdots + q_{r'}(x)f^{(r')}(x) = R(x), \tag{9}$$

where the order r' is at most k, the degree of the coefficient $q_t(x)$ for each t is at most $r+k$, and the degree of the polynomial $R(x)$ is at most $r-1$.

Definition. The generating function $f(x)$ of a holonomic sequence is called *holonomic*. Also, $f(x)$ satisfying (9) is called *D-finite* (differentiably finite), see [14].

Proof. Assume that a_n satisfies the relation

$$p_r(n)a_{n+r} + p_{r-1}(n)a_{n+r-1} + \cdots + p_0(n)a_n = 0.$$

We denote by $b_{s,t}$ the coefficient of $a_{n+t}n^s$ in the relation above, that is, $p_t(n) = \sum_{s=0}^{k} b_{s,t}n^s$, and so the holonomic relation becomes

$$\sum_{t=0}^{r}\sum_{s=0}^{k} b_{s,t}n^s a_{n+t} = 0. \tag{10}$$

To prove our results, first we note the following identity. For fixed s and t,

$$\begin{aligned}\sum_{j=0}^{s} c_j^{(s,t)} x^{j-t} f^{(j)}(x) &= \sum_{j=0}^{s} c_j^{(s,t)} \sum_{n=0}^{\infty} (n)_j a_n x^{n-t} \\ &= \sum_{j=0}^{s} c_j^{(s,t)} \sum_{n=-t}^{\infty} (n+t)_j a_{n+t} x^n \\ &= \sum_{n=-t}^{\infty} \left[\sum_{j=0}^{s} c_j^{(s,t)} (n+t)_j \right] a_{n+t} x^n,\end{aligned}$$

where $(n)_j$ denotes the falling factorial, i.e. $(n)_j = n(n-1)\cdots(n-j+1)$, and $c_j^{(s,t)}$'s are some constants.
For each pair of (s,t), we appeal to the method of equating the coefficients to obtain $c_j^{(s,t)}$, $j = 0, 1, 2, \ldots, s$. Equating the corresponding coefficients of n^j in the equation $\sum_{j=0}^{s} c_j^{(s,t)}(n+t)_j = n^s$ results in the system of $s+1$ linear equations with $s+1$ unknowns, and so the unknown constants $c_j^{(s,t)}$ can be determined.
Next, we define $A_{s,t}(x) = \sum_{n=0}^{\infty} a_{n+t}n^s x^n$ for $s, t \geq 0$. Then,

$$\sum_{j=0}^{s} c_j^{(s,t)} x^{j-t} f^{(j)}(x) = \sum_{n=-t}^{\infty} a_{n+t}n^s x^n = A_{s,t}(x) + \sum_{n=-t}^{-1} a_{n+t}n^s x^n. \tag{11}$$

From the holonomic relation of a_n in (10), multiply x^n through and sum n from 0 to ∞:

$$\begin{aligned}0 &= \sum_{n=0}^{\infty}\sum_{t=0}^{r}\sum_{s=0}^{k} b_{s,t}n^s a_{n+t}x^n = \sum_{t=0}^{r}\sum_{s=0}^{k} b_{s,t}A_{s,t}(x) \\ &= \sum_{t=0}^{r}\sum_{s=0}^{k} b_{s,t}\left[\sum_{j=0}^{s} c_j^{(s,t)} x^{j-t} f^{(j)}(x) - \sum_{n=-t}^{-1} a_{n+t}n^s x^n\right], \quad \text{from (11)}.\end{aligned}$$

Multiplying x^r on both sides and rearranging this equation, we obtain

$$\sum_{t=0}^{r}\sum_{s=0}^{k} b_{s,t}\sum_{j=0}^{s} c_j^{(s,t)} x^{j+r-t} f^{(j)}(x) = \sum_{t=0}^{r}\sum_{s=0}^{k} b_{s,t}\sum_{n=0}^{t-1} a_n(n-t)^s x^{n+r-t}.$$

Observe that the left hand side is the differential equation of order at most k and degree at most $r+k$, while the right hand side is the polynomial of degree at most $r-1$ as desired. □

Example 1: Let $a_n = n!$. Then, a_n satisfies the holonomic recurrence

$$a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1.$$

The differential equation corresponding to its generating function is

$$(1-x)f(x) - x^2 f'(x) = 1.$$

Maple Program

```
> HoToDiff(n+1-N,[1],n,N,x,D);
```

Example 2: Let $a_n = \dfrac{1}{n+1}\dbinom{2n}{n}$, which satisfies the holonomic recurrence

$$(4n+2)a_n - (n+2)a_{n+1} = 0.$$

The differential equation corresponding to its generating function is

$$(1-2x)f(x) + (x-4x^2)f'(x) = 1.$$

It is well-known that a closed-form expression for the generating function $f(x)$ is $\dfrac{1-\sqrt{1-4x}}{2x}$. The reader can easily verify that this expression of $f(x)$ satisfies the above differential equation.

Maple Program

```
> HoToDiff(4*n+2-(n+2)*N,[1],n,N,x,D);
```

Example 3: Let $a_n = \left\lfloor \left(\dfrac{n}{2}\right)^2 \right\rfloor$ with a holonomic recurrence given by

$$(n+2)a_n + 2a_{n+1} - na_{n+2} = 0.$$

The differential equation corresponding to its generating function is

$$2(1+x+x^2)f(x) + (-x+x^3)f'(x) = 0.$$

On the other hand, another holonomic (C-finite) recurrence of a_n is

$$a_{n+4} - 2a_{n+3} + 2a_{n+1} - a_n = 0,$$

which leads to the (zero-order) differential equation relation

$$(1+x)(1-x)^3 f(x) = x^2.$$

The reader is encouraged to check that $f(x)$ from the zero-order relation satisfies the first-order differential equation.

Maple Program

```
> HoToDiff(n+2+2*N-N^2*n,[0,0],n,N,x,D);
> HoToDiff(1-2*N+2*N^3-N^4,[0,0,1,2],n,N,x,D);
```

A non-homogeneous differential equation in (9) can be transformed into a homogeneous one by performing the following steps. First, we write the non-homogeneous relation $L(x, D) = R(x)$. Next, differentiating both sides of the equation w.r.t. x, we get $L'(x, D) = R'(x)$. This leads to the sought after homogeneous relation:

$$R'(x) \cdot L(x, D) - R(x) \cdot L'(x, D) = 0.$$

We state this fact in the following corollary.

Corollary 13. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where a_n is a holonomic sequence of order r and degree k. Then, $f(x)$ satisfies a homogeneous linear differential equation with polynomial coefficients,*

$$q_0(x)f(x) + q_1(x)f'(x) + \cdots + q_{r'}(x)f^{(r')}(x) = 0, \tag{12}$$

where the order r' is at most $k+1$, and the degree of $q_t(x)$ for each t is at most $2r + k - 1$.

Remark. Theorem 7.1.2 of [15] appears to contain a typographical error in the bounds of the order r' and the degree of $q_t(x)$ of the homogeneous linear differential equation (12), having k and $r+k$ specified as the bounds therein as opposed to $k+1$ and $2r+k-1$, respectively.

Example: The homogeneous differential equation of $f(x) = \sum_{n=0}^{\infty} n! x^n$ is

$$x^2 f''(x) + (3x - 1)f'(x) + f(x) = 0.$$

Maple Program

```
> HoToDiffHom(n+1-N,[1],n,N,x,D);
```

The next theorem, which is the converse of Corollary 13, ensures that one can always establish a holonomic recurrence relation for the coefficients a_n of $f(x)$ satisfying a homogeneous linear differential equation with polynomial coefficients.

Theorem 14. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Assume $f(x)$ satisfies a homogeneous linear differential equation with polynomial coefficients of order r and degree k,*

$$q_0(x)f(x) + q_1(x)f'(x) + \cdots + q_r(x)f^{(r)}(x) = 0. \tag{13}$$

Then, a_n is a holonomic sequence of order at most $r+k$ and degree at most r.

Proof. For $0 \le t \le r$, let $b_{s,t}$ be the coefficient of $x^s f^{(t)}(x)$ so that $q_t(x) = \sum_{s=0}^{k} b_{s,t} x^s$. We note that, for each s and t,

$$x^s f^{(t)}(x) = \sum_{n=t}^{\infty} (n)_t a_n x^{n-t+s} = \sum_{n=s}^{\infty} (n+t-s)_t a_{n+t-s} x^n.$$

Now (13) can be written as

$$\sum_{t=0}^{r} \sum_{s=0}^{k} \sum_{n=s}^{\infty} b_{s,t} (n+t-s)_t a_{n+t-s} x^n = 0.$$

Then, for each $n \ge 0$, a_n satisfies the following recurrence

$$\sum_{t=0}^{r} \sum_{s=0}^{\min(n,k)} b_{s,t} (n+t-s)_t a_{n+t-s} = 0,$$

and the claim about the bounds on the order and degree follows immediately. □

5. Closure properties:

Theorem 15. *Assume that a_n and b_n are holonomic sequences of order r and s, respectively. The following sequences are also holonomic sequences, with the specified upper bound on the order.*

(i) addition: $(a_n + b_n)_{n=0}^{\infty}$, of order at most $r + s$,
(ii) term-wise multiplication: $(a_n \cdot b_n)_{n=0}^{\infty}$, of order at most rs,
(iii) partial sum: $(\sum_{i=0}^{n} a_i)_{n=0}^{\infty}$, of order at most $r + 1$,
(iv) linear subsequence: $(a_{mn})_{n=0}^{\infty}$, where m is a positive integer, of order at most r.

Furthermore, let $c_n = \sum_{i=0}^{n} a_i \cdot b_{n-i}$ be the Cauchy product of a_n and b_n. Assume that the generating functions of a_n and b_n, denoted by $A(x)$ and $B(x)$, satisfy homogeneous differential equations of orders r_1 and r_2, respectively. Then, the generating function of c_n also satisfies a homogeneous differential equation of order at most $r_1 r_2$.

Let us reiterate here that if the resulting sequence under these operations has a zero leading coefficient $p_r(n)$ for some n, then the recurrence relation is only valid for $n > n_0$ where n_0 is the largest positive integer root of the equation $p_r(n) = 0$.

Proof. To verify the closure properties *(i)*-*(iv)*, we follow the solution space approach in the same vein as for C-finite sequences. As for the Cauchy product, it is worth pointing out that while the proof also relies on the solution space approach, this time we work with the generating function instead of the sequence itself.

We first give the proof of *(i)*-*(iv)*. Assume a_n is a holonomic sequence of order r, i.e.

$$p_r(n)a_{n+r} + p_{r-1}(n)a_{n+r-1} + \cdots + p_0(n)a_n = 0,$$

and b_n is a holonomic sequence of order s, i.e.

$$q_s(n)b_{n+s} + q_{s-1}(n)b_{n+s-1} + \cdots + q_0(n)b_n = 0.$$

We note that for each fixed k, $k \geq r$, a_{n+k} can be written as a linear combination, with rational function coefficients, of $a_{n+r-1}, a_{n+r-2}, \ldots, a_n$. This can be seen by repeated substitution starting from the term a_{n+r}, $a_{n+r+1}, \ldots,$ all the way to a_{n+k}. The same argument can be made for $b_{n+k}, k \geq s$.

- addition: let $c_n = a_n + b_n$. Then $c_n, c_{n+1}, \ldots, c_{n+r+s}$ can be put in the system of linear equations as

$$\begin{bmatrix} c_n \\ c_{n+1} \\ c_{n+2} \\ \cdots \\ c_{n+r+s-1} \\ c_{n+r+s} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \ldots\ 1\ 0\ 0 \ldots \\ 0 & 1 & 0 & \ldots\ 0\ 1\ 0 \ldots \\ & & & \ldots \\ M^{(1)}_{r+s-1}(n) & M^{(2)}_{r+s-1}(n) & M^{(3)}_{r+s-1}(n) & \ldots \\ M^{(1)}_{r+s}(n) & M^{(2)}_{r+s}(n) & M^{(3)}_{r+s}(n) & \ldots \end{bmatrix} \cdot \begin{bmatrix} a_n \\ a_{n+1} \\ \cdots \\ a_{n+r-1} \\ b_n \\ b_{n+1} \\ \cdots \\ b_{n+s-1} \end{bmatrix}.$$

  Since the matrix M with rational entries (rational matrix) in the middle has $r+s+1$ rows and $r+s$ columns, its null space is non-trivial, i.e. there exists a row vector $P \neq 0$ such that $P \cdot M = [0, 0, \ldots, 0]$. Note, in addition, that the solution P in the form of rational functions has one free variable (as the number of rows > number of columns). We can turn the solutions in P to polynomials. This P gives a holonomic relation to $c_n, c_{n+1}, \ldots, c_{n+r+s}$.
- term-wise multiplication: In a similar way as in the addition case, let $c_n = a_n \cdot b_n$. Then $c_n, c_{n+1}, \ldots, c_{n+rs}$ can be put in the system of linear equations as

$$\begin{bmatrix} c_n \\ c_{n+1} \\ c_{n+2} \\ \cdots \\ c_{n+rs-1} \\ c_{n+rs} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \ldots\ 0\ 0\ 0 \ldots \\ 0 & 0 & 0 & \ldots\ 0\ 1\ 0 \ldots \\ & & & \ldots \\ M^{(1)}_{rs-1}(n) & M^{(2)}_{rs-1}(n) & M^{(3)}_{rs-1}(n) & \ldots \\ M^{(1)}_{rs}(n) & M^{(2)}_{rs}(n) & M^{(3)}_{rs}(n) & \ldots \end{bmatrix} \cdot \begin{bmatrix} a_n b_n \\ a_n b_{n+1} \\ \cdots \\ a_{n+1} b_n \\ a_{n+1} b_{n+1} \\ \cdots \\ a_{n+r-1} b_{n+s-1} \end{bmatrix}.$$

  Again since the matrix M has $rs + 1$ rows and rs columns, the null space of M is non-trivial. A polynomial solution $P \neq 0$ gives a holonomic relation to $c_n, c_{n+1}, \ldots, c_{n+rs}$.

- linear subsequence: For a fixed integer m, let $c_n = a_{mn}$. Then, $c_n, c_{n+1}, \dots, c_{n+r}$ can be put in the system of linear equations as

$$\begin{bmatrix} c_n \\ c_{n+1} \\ c_{n+2} \\ \dots \\ c_{n+r} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ & & & \dots & \\ M_{r-1}^{(1)}(mn) & M_{r-1}^{(2)}(mn) & M_{r-1}^{(3)}(mn) & \dots & M_{r-1}^{(r)}(mn) \\ M_r^{(1)}(mn) & M_r^{(2)}(mn) & M_r^{(3)}(mn) & \dots & M_r^{(r)}(mn) \end{bmatrix} \cdot \begin{bmatrix} a_{mn} \\ a_{mn+1} \\ a_{mn+2} \\ \dots \\ a_{mn+r-1} \end{bmatrix}.$$

Here, the matrix M has $r+1$ rows and r columns, so a non-trivial polynomial solution P (in a null space of M) exists. This P gives a holonomic relation to $c_n, c_{n+1}, \dots, c_{n+r}$.
- partial sum: We set up a slightly different matrix equation this time to simplify the computation. Let $c_n = \sum_{i=0}^{n} a_i$. Then, $c_{n-1}, c_n, c_{n+1}, \dots, c_{n+r}$ satisfy the following system of linear equations

$$\begin{bmatrix} c_n - c_{n-1} \\ c_{n+1} - c_n \\ c_{n+2} - c_{n+1} \\ \dots \\ c_{n+r} - c_{n+r-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ -\frac{p_0(n)}{p_r(n)} & -\frac{p_1(n)}{p_r(n)} & -\frac{p_2(n)}{p_r(n)} & \dots & -\frac{p_{r-1}(n)}{p_r(n)} \end{bmatrix} \cdot \begin{bmatrix} a_n \\ a_{n+1} \\ \dots \\ a_{n+r-2} \\ a_{n+r-1} \end{bmatrix}.$$

The matrix M has $r+1$ rows and r columns, so a non-trivial polynomial solution P exists in the null space of M. This P gives a holonomic relation of order $r+1$ to $c_{n-1}, c_n, c_{n+1}, \dots, c_{n+r}$.

We now turn to the proof for the Cauchy product.
- Cauchy product: Let $c_n = \sum_{i=0}^{n} a_i \cdot b_{n-i}$. This time, we consider finding the homogeneous differential equation of $C(x) = \sum_{n=0}^{\infty} c_n x^n$. Then we can invoke Theorem 14, the relationship between $C(x)$ and the c_n itself, to conclude that c_n is a holonomic sequence.

To this end, let us express the homogeneous differential equation of $A(x)$ as

$$p_0(x)A(x) + p_1(x)A'(x) + \cdots + p_{r_1}(x)A^{(r_1)}(x) = 0,$$

and the homogeneous differential equation of $B(x)$ as

$$q_0(x)B(x) + q_1(x)B'(x) + \cdots + q_{r_2}(x)B^{(r_2)}(x) = 0.$$

Note that $C(x) = A(x) \cdot B(x)$ and that any order derivatives of $C(x)$ can be written as a linear combination of $A^{(i)}(x) \cdot B^{(j)}(x), \quad 0 \le i \le r_1 - 1$ and $0 \le j \le r_2 - 1$. Hence, we can write the relation in matrix notation as:

$$\begin{bmatrix} C(x) \\ C'(x) \\ C''(x) \\ \dots \\ C^{(r_1r_2-1)}(x) \\ C^{(r_1r_2)}(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots 0\ 0\ 0 \dots \\ 0 & 1 & 0 & \dots 1\ 0\ 0 \dots \\ & & & \dots \\ M^{(1)}_{r_1r_2-1}(n) & M^{(2)}_{r_1r_2-1}(n) & M^{(3)}_{r_1r_2-1}(n) & \dots \\ M^{(1)}_{r_1r_2}(n) & M^{(2)}_{r_1r_2}(n) & M^{(3)}_{r_1r_2}(n) & \dots \end{bmatrix} \cdot \begin{bmatrix} A(x)B(x) \\ A(x)B'(x) \\ \dots \\ A'(x)B(x) \\ A'(x)B'(x) \\ \dots \\ A^{(r_1-1)}(x)B^{(r_2-1)}(x) \end{bmatrix}.$$

Using the same arguments as above, since the matrix M has r_1r_2+1 rows and r_1r_2 columns, the existence of a non-trivial polynomial solution P (in the null space of M) is ensured. This P gives a homogeneous relation in terms of the differential equation of order r_1r_2 to $C(x), C'(x), \dots, C^{(r_1r_2)}(x)$. □

Corollary 16. *Let* $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *be holonomic. Then* $A'(x)$ *and* $\int A(x)$ *are also holonomic.*

Proof. $A'(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$. Since $n+1$ and a_{n+1} are holonomic sequences, by the closure property of multiplication, so is the sequence $(n+1)a_{n+1}$. Hence, $A'(x)$ is holonomic. A similar argument holds for $\int A(x)$. □

Example: Closure properties

Let $a_n = \dfrac{1}{n+1}\binom{2n}{n}$, the Catalan numbers. Recall that a_n satisfies the holonomic recurrence of order 1 and degree 1:

$$(4n+2)a_n - (n+2)a_{n+1} = 0.$$

Let $b_n = \sum_{i=1}^{n} \dfrac{1}{i}$, the harmonic numbers. Here, b_n satisfies the holonomic recurrence of order 2 and degree 1:

$$(n+1)b_n - (2n+3)b_{n+1} + (n+2)b_{n+2} = 0.$$

We first show detailed calculations for the closure properties of addition and term-wise multiplication.

- addition: $c_n = a_n + b_n$. Consider the matrix equation:

$$\begin{bmatrix} c_n \\ c_{n+1} \\ c_{n+2} \\ c_{n+3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ \frac{2(2n+1)}{n+2} & 0 & 1 \\ \frac{4(3+2n)(1+2n)}{(3+n)(2+n)} & -\frac{n+1}{n+2} & \frac{2n+3}{n+2} \\ \frac{8(5+2n)(3+2n)(1+2n)}{(4+n)(3+n)(2+n)} & -\frac{(5+2n)(n+1)}{(3+n)(2+n)} & \frac{(12n+3n^2+11)}{(3+n)(2+n)} \end{bmatrix} \cdot \begin{bmatrix} a_n \\ b_n \\ b_{n+1} \end{bmatrix}.$$

The polynomial solution P in the null space of the rational matrix M is

$$P = [-2(n+1)(3n+7)(1+2n)(2+n)^2, (5+3n)(2+n)(9n^3+43n^2+58n+20), \dots].$$

This gives rise to the holonomic relation of c_n of order 3 as

$$\begin{aligned}
&-2(n+1)(3n+7)(1+2n)(2+n)^2 c_n + (5+3n)(2+n)(9n^3+43n^2+58n+20)c_{n+1}\\
&-(216n+241n^2+111n^3+64+18n^4)(3+n)c_{n+2}\\
&+(3+n)(4+n)(3n+4)(n+1)^2 c_{n+3} = 0.
\end{aligned}$$

Maple Program

```
> R1:= 2+4*n+(-2-n)*N:
> R2:= n+1+(-3-2*n)*N+(2+n)*N^2:
> HoAdd(R1,R2,n,N,c);
```

- term-wise multiplication: $d_n = a_n \cdot b_n$. Consider

$$\begin{bmatrix} d_n \\ d_{n+1} \\ d_{n+2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{2(2n+1)}{n+2} \\ \frac{4(3+2n)(1+2n)(-1-n)}{(3+n)(2+n)^2} & \frac{4(3+2n)^2(1+2n)}{(3+n)(2+n)^2} \end{bmatrix} \cdot \begin{bmatrix} a_n b_n \\ a_n b_{n+1} \end{bmatrix}.$$

The polynomial solution P in the null space of the rational matrix M is

$$P = [4(3+2n)(1+2n)(n+1), -2(3+2n)^2(2+n), (2+n)^2(3+n)].$$

This gives rise to the holonomic relation of d_n of order 2 as

$$4(3+2n)(1+2n)(n+1)d_n - 2(3+2n)^2(2+n)d_{n+1} + (2+n)^2(3+n)d_{n+2} = 0.$$

Maple Program

```
> R1:= 2+4*n+(-2-n)*N:
> R2:= n+1+(-3-2*n)*N+(2+n)*N^2:
> HoTermWise(R1,R2,n,N,c);
```

We now give an example that shows how to obtain a homogeneous differential equation for the Cauchy product.

- Cauchy product: $e_n = \sum_{i=0}^{n} a_i \cdot b_{n-i}$.

In this example, we let $a_n = \dfrac{1}{n+1}\binom{2n}{n}$ and $b_n = n!$.
As a_n and b_n are holonomic, the homogeneous differential equations $A(x)$ and $B(x)$ exist. Indeed, they satisfy

$$\begin{aligned}
x(4x-1)A''(x) + 2(5x-1)A'(x) + 2A(x) = 0,\\
x^2B''(x) + (3x-1)B'(x) + B(x) = 0.
\end{aligned}$$

Letting $E(x) = A(x) \cdot B(x)$, we obtain the matrix equation

$$\begin{bmatrix} E(x) \\ E'(x) \\ E''(x) \\ E^{(3)}(x) \\ E^{(4)}(x) \end{bmatrix} = \begin{bmatrix} & 1 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 0 \\ -\frac{1}{(x^2)} & -\frac{2}{x(4x-1)} & -\frac{(3x-1)}{x^2} & -\frac{2(5x-1)}{x(4x-1)} & 2 \\ & & \cdots & & \\ & & \cdots & & \end{bmatrix} \cdot \begin{bmatrix} A(x)B(x) \\ A(x)B'(x) \\ A'(x)B(x) \\ A'(x)B'(x) \end{bmatrix}.$$

The polynomial solution P in the null space of the rational matrix M is found to be

$$P = [2(72x^6 + 660x^5 - 1392x^4 + 900x^3 - 266x^2 + 37x - 2)(4x - 1), \ldots].$$

This gives rise to the homogeneous differential equation of $C(x)$ of order $2 \cdot 2 = 4$ as

$$\begin{aligned} &2(72x^6 + 660x^5 - 1392x^4 + 900x^3 - 266x^2 + 37x - 2)(4x - 1)E(x) \\ &+ 2(1512x^8 + 11076x^7 - 26812x^6 + 22170x^5 - 9442x^4 + 2333x^3 - 342x^2 + 28x - 1)E'(x) \\ &+ \cdots + x^5(4x - 1)^2(4x^4 + 24x^3 - 31x^2 + 10x - 1)E^{(4)}(x) = 0. \end{aligned}$$

Maple Program

```
> DA:= lhs(HoToDiffHom(4*n+2-(n+2)*N,[1],n,N,x,
D))/f(x);
> DB:= lhs(HoToDiffHom(n+1-N,[1],n,N,x,D))
/f(x);
> HoCauchy(DA,DB,x,D,c);
```

A remark on the rigorous proof of identities and the upper bound for the degree of recurrence relations

As already mentioned previously in the C-finite section, the closure property is utterly important. Theorem 15 gives bounds for the order of recurrence relation. Meanwhile, bounds for the degree of polynomial coefficients can also be determined. Once the bounds for the order and degree are known, if the holonomic ansatz is fit with enough data, a rigorous recurrence relation can be ensured.

We now illustrate how we can obtain a good pragmatic upper bound for the degree using the following example.

Example: Let us try to figure out an upper bound of the degree of the recurrence relation under the addition: $c_n = a_n + b_n$. Using the same example as above, recall that the bound for the order of c_n is $r = 2+1 = 3$, and the following rational matrix was obtained in the intermediate steps:

$$M = \begin{bmatrix} 1 & 1 & 0 \\ \frac{2(2n+1)}{n+2} & 0 & 1 \\ \frac{4(3+2n)(1+2n)}{(3+n)(2+n)} & -\frac{n+1}{n+2} & \frac{2n+3}{n+2} \\ \frac{8(5+2n)(3+2n)(1+2n)}{(4+n)(3+n)(2+n)} & -\frac{(5+2n)(n+1)}{(3+n)(2+n)} & \frac{(12n+3n^2+11)}{(3+n)(2+n)} \end{bmatrix}.$$

Getting rid of the denominator, we obtain the matrix

$$\tilde{M} = \begin{bmatrix} (4+n)(3+n)(2+n) & (3+n)(2+n) & 0 \\ 2(2n+1)(4+n)(3+n) & 0 & (3+n)(2+n) \\ 4(3+2n)(1+2n)(4+n) & -(n+1)(3+n) & (2n+3)(3+n) \\ 8(5+2n)(3+2n)(1+2n) & -(5+2n)(n+1) & 12n+3n^2+11 \end{bmatrix}$$

whose polynomial entries have the highest degree of $v = 3$.
Finding a bound for the degree of c_n amounts to determining the degree of the polynomial solution $P = [p_1(n), p_2(n), \cdots, p_{r+1}(n)]$ in the null space of $\tilde{M}$.
Formally, let us assume that the highest degree of polynomial in P is k. Then, $p_l(n) = \sum_{j=0}^{k} b_{j,l} n^j$, $l = 1, 2, \ldots, r+1$. By the method of equating the coefficients, the matrix equation $P \cdot \tilde{M} = 0$ results in the system of $r(k+1+v)$ linear equations with $(r+1)(k+1)$ unknowns. The existence of a solution to this linear system is guaranteed whenever k satisfies the inequality $(r+1)(k+1) > r(k+1+v)$, i.e. $k \geq rv$. Therefore, this condition gives rise to a pragmatic upper bound for k, the degree of the holonomic sequence c_n.

6. **Asymptotic approximation solutions:**
Unlike the C-finite recurrences, in general, no closed form solution is available for holonomic recurrences. Hence, an approximation solution in terms of asymptotic expansion will be sought for a holonomic recurrence. As the method is quite complicated, we will not attempt to provide a theoretical analysis, but rather some applications of the method. For a more thorough account on the subject, the interested reader is referred to [17]. In what follows, the sequence a_n will be treated as a function. To emphasize this fact we will denote it by $y(n)$.

Suppose that $y(n)$ is a solution to

$$\sum_{i=0}^{r} p_i(n) y(n+i) = 0, \tag{14}$$

where $p_r(n) \neq 0, \quad n = 0, 1, 2, \ldots$.
The approach is based on the Birkhoff-Trjitzinsky method [1,3]. Although the detailed analysis of the method was given in [1,3], we adopt here a variant which assumes a solution in its simplest form of (15). Then, substitute the assumed solution into (14) (with the help of (16)) to find values for the parameters. For this reason, we will refer to the method as

the *guess and check* in this paper. Despite its simplicity, this variant has proven to be applicable to a large class of holonomic recurrences.

Guess and check method:
The guess and check method is a general method of solving holonomic recurrences. What we will do is to try a simple solution of the form

$$y(n) = e^{\mu_0 n \ln n + \mu_1 n} \cdot n^{\theta} \cdot e^{\alpha_1 n^{\beta} + \alpha_2 n^{\beta - \frac{1}{\rho}} + \alpha_3 n^{\beta - \frac{2}{\rho}} + \dots}, \tag{15}$$

where the parameters μ_0, μ_1, θ are complex, $\rho\,(\rho \geq 1), \mu_0\rho$ are integers, and $\alpha_1 \neq 0, \quad \beta = \frac{j}{\rho}, \, 0 \leq j < \rho$.
This method provides r independent solutions (all formal series solutions) but only up to constant multiple. The function in the form of (15) which satisfies (14) is called *a formal series solution* (FSS).
Using some power series expansion and simplification, we obtain

$$\frac{y(n+k)}{y(n)} = n^{\mu_0 k} \lambda^k \cdot \{1 + \frac{1}{n}\left(\theta k + \frac{k^2 \mu_0}{2}\right) + \dots\} \cdot e^{\alpha_1 \beta k n^{\beta-1} + \alpha_2 (\beta - \frac{1}{\rho}) k n^{\beta - \frac{1}{\rho} - 1} + \dots}, \tag{16}$$

for $k \geq 0$, where $\lambda = e^{\mu_0 + \mu_1}$.

Applications:
Let us give walkthrough examples to demonstrate the approach. Since the procedure involves some steps that require human input and expertise, no Maple program is provided in this section.

Example 1: $y(n) = n!$. The most standard and widely used asymptotic formula for the factorial function is Stirling's formula. In this example, we will try to obtain an asymptotic approximation for the factorial function using the method of guess and check.
From the recurrence relation $y(n+1) - (n+1)y(n) = 0$, we apply (16):

$$n^{\mu_0} \lambda \{1 + \frac{1}{n}\left(\theta + \frac{\mu_0}{2}\right) + \dots\} e^{\alpha_1 \beta n^{\beta - 1} + \dots} - (n+1) = 0.$$

Expanding the exponential term with power series and comparing the terms involving n^{μ_0} and n, we have $\mu_0 = 1$, $\lambda = 1$. Also, since $\mu_0\rho$ must be an integer, we assign $\rho = 1$, the minimum possible value.
Next, the value of β must be 0, as the coefficient of n^s for $0 < s < 1$ must be 0.
For the coefficient of 1 (the constant term), we have $\theta + \dfrac{\mu_0}{2} - 1 = 0$. Hence, $\theta = \dfrac{1}{2}$.
Plugging in these parameters back to (15), we arrive at

$$y(n) = K \left(\frac{n}{e}\right)^n \sqrt{n} \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \dots\right).$$

We note that the infinite series on the right most arises from the series expansion of $e^{\alpha_2 n^{-1} + \alpha_3 n^{-2} + \dots}$. The values of $c_1, c_2, \dots$ can be figured out by plugging $y(n)$ back into the original recurrence and comparing the coefficient of n^i for each i (the method of undetermined coefficients). The constant K, however, cannot be obtained by this method although another asymptotic method shows that $K = \sqrt{2\pi}$.

Example 2: Let $y(n) = \left\lfloor \left(\frac{n}{2}\right)^2 \right\rfloor$ with the holonomic recurrence

$$(n+2)y(n) + 2y(n+1) - ny(n+2) = 0.$$

Apply (16) to get the relation:

$$(n+2) + 2n^{\mu_0}\lambda\{1 + \frac{1}{n}\left(\theta + \frac{\mu_0}{2}\right) + \dots\}e^{\alpha_1\beta n^{\beta-1}+\dots} \\ - nn^{2\mu_0}\lambda^2\{1 + \frac{1}{n}(2\theta + 2\mu_0) + \dots\}e^{2\alpha_1\beta n^{\beta-1}+\dots} = 0.$$

Expanding the exponential term with power series and comparing the terms involving $n^{2\mu_0}$ and n, we have $\mu_0 = 0$. Then, $1 - \lambda^2 = 0$, which gives $\lambda = \pm 1$. Also, since $\mu_0\rho$ must be an integer, we again assign the minimum possible value $\rho = 1$.
Next, β must be 0, as the coefficient of n^s for $0 < s < 1$ must be 0.
For the coefficient of 1 (the constant term), we have $2+2\lambda-\lambda^2(2\theta+2\mu_0) = 0$. Hence, $\theta = \frac{1+\lambda}{\lambda^2} = 2$ or 0.
Plugging these parameters back into (15), we obtain

$$y_1(n) = K_1 n^2\left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \dots\right),$$

$$y_2(n) = K_2(-1)^n\left(1 + \frac{d_1}{n} + \frac{d_2}{n^2} + \dots\right),$$

where $y(n) = y_1(n) + y_2(n)$.
With this form of solution, we apply the method of undetermined coefficients to the original recurrence, which in turn implies that c_i and d_i are all zero except $c_2 = -1/2$. Hence, $y(n) = K_1 n^2 - \frac{K_1}{2} + K_2(-1)^n$, agreeing with our earlier result that $y(n) = \frac{n^2}{4} - \frac{1}{8} + \frac{(-1)^n}{8}$.

3 C^2-finite

Building upon C-finite, the class of C^2-finite sequences that we will investigate in this section is a relatively new domain of research. The idea was first mentioned in [10] in the context of graph polynomials, and it has recently been discussed in [5,16] from a theoretical perspective.

1. **Ansatz:** a_n is defined by a linear recurrence with C-finite sequence coefficients:
$$C_{r,n}a_{n+r} + C_{r-1,n}a_{n+r-1} + \cdots + C_{0,n}a_n = 0,$$
where $C_{r,n} \neq 0$, $n = 0, 1, 2, \ldots$, along with the initial values $a_0, a_1, \ldots, a_{r-1}$. We call a_n a C^2*-finite sequence* of order r. This term was first coined in [10].
As was the case for holonomic sequences, the condition $C_{r,n} \neq 0$ is necessary for recursively computing the value of a_{n+r} from preceding terms.

2. **Example 1:** C^2-finite sequence of order 1 given by
$$a_n = F_{n+1} \cdot a_{n-1}, \qquad a_0 = 1,$$
where F_n is the Fibonacci sequence.
An interesting fact about this sequence is that it also satisfies a non-linear relationship
$$a_n a_{n+1} a_{n+3} - a_n a_{n+2}^2 - a_{n+2} a_{n+1}^2 = 0.$$
This nonlinear relation was provided by Robert Israel/Michael Somos in 2014. It is the sequence A003266 on the Sloane's OEIS, [12].
This example motivated us to investigate connections between the class of C^2-finite sequences and a non-linear recurrence representation. Especially, since the conditions which ensure the existence of nonlinear recurrences for the C-finite (sub)sequences have already been examined in [4], we are curious to know if similar results might be obtained for C^2-finite sequences. Furthermore, it is also unclear whether or not one can find conditions for which a non-linear recurrence is a C^2-finite sequence. We leave these as open problems for the interested readers.
Open problem 1: Find conditions which guarantee that a C^2-finite sequence can be represented in a non-linear recurrence relation.
Open problem 2: Find conditions which guarantee that a non-linear recurrence is a C^2-finite sequence.

   Example 2: C^2-finite sequence of order 2 given by
$$a_{n+2} = a_{n+1} + 2^n a_n,$$
with initial values $a_0 = 1, a_1 = 1$. In terms of the shift operator,
$$[N^2 - N - 2^n] \cdot a_n = 0.$$

   Example 3: C^2-finite sequence of order 2 given by
$$a_{n+2} = F_{n+1}a_{n+1} + F_n a_n,$$
with initial values $a_0 = 1, a_1 = 1$.

3. **Guessing:**

INPUT: the order r of a_n, the order d for each C-finite coefficient $C_{i,n}$, $0 \le i \le r$, and a sufficiently long sequence of data a_n.

This time guessing becomes very difficult due to the challenge we face when solving a system of nonlinear equations. We illustrate this by the following examples.
The simplest, non-trivial example is the second order relation where $C_{1,n}$ and $C_{0,n}$ are of first order. The form of ansatz is

$$a_{n+2} = c_1\alpha^n a_{n+1} + c_2\beta^n a_n, \quad n \ge 0.$$

Here, we solve for constants $\{\alpha, \beta, c_1, c_2\}$ through the system of nonlinear equations. With four parameters, Maple can still handle the computation in this case.
However, if the second order relation is assumed with $C_{1,n}$ and $C_{0,n}$ of second order, the guessing is much more complicated. For example, if we assume

$$a_{n+2} = (c_1\alpha_1^n + c_2\alpha_2^n)a_{n+1} + (c_3\alpha_3^n + c_4\alpha_4^n)a_n, \quad n \ge 0,$$

with eight parameters to solve for, this time the problem becomes computationally infeasible.
Another approach that could be useful for guessing a C^2-finite relation is to apply a numerical solution method. In Maple, we can do this with the available `fsolve` built-in command. Unfortunately, even with the numerical method, we were not able to obtain the solution within a finite number of steps. It was rather disappointing to find that guessing for C^2-finite is not practical, as it plays a big role in determining an expression for the sequences.

4. Generating function:

In this section, we establish several new properties for C^2-finite sequences. First, we give a formal definition of C^2-finite.
We recall from the C-finite section that a closed-form formula for a C-finite sequence C_n is

$$C_n = \sum_{\alpha \in S} p_\alpha(n)\alpha^n,$$

where α's are the roots of the characteristic polynomial of C_n.
We define $Deg(C_n)$ to be the highest degree of $p_\alpha(n)$, $\alpha \in S$.

Definition. A C^2-finite sequence a_n is said to have order r and degree k if a_n satisfies the recurrence relation

$$C_{r,n}a_{n+r} + C_{r-1,n}a_{n+r-1} + \cdots + C_{0,n}a_n = 0,$$

where for each i, $0 \le i \le r$, $Deg(C_{i,n})$ is at most k.

We are now ready to derive a new differential equation for the generating function of C^2-finite sequence. This inquiry was made in [5].

Theorem 17. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *where* a_n *is a* C^2*-finite sequence of order* r *and degree* k*. Then,* $f(x)$ *satisfies a (non-homogeneous) linear differential equation with polynomial coefficients,*

$$\sum_{\alpha} \left[q_{\alpha,0}(x) f(\alpha x) + q_{\alpha,1}(x) f'(\alpha x) + \cdots + q_{\alpha,r'}(x) f^{(r')}(\alpha x) \right] = R(x) \tag{17}$$

where α *is defined to be* $\alpha_{i,j}$*, the root of the characteristic polynomial of* $C_{i,n}$*. Here, the order* r' *is at most* k*, degree of* $q_{\alpha,t}(x)$ *for each* α *and* t *is at most* $r+k$*, and the degree of polynomial* $R(x)$ *is at most* $r-1$*.*

Notation. To avoid any ambiguity in our notation $f^{(j)}(\alpha x)$, this notation means

$$f^{(j)}(\alpha x) = \frac{d^j f(\alpha x)}{dx^j}.$$

Definition. The function $f(x)$ as a generating function of a C^2-finite sequence is called *C*^{2}*-finite*. Also, we will call a function $f(x)$ that satisfies (17) *DC-finite* (differentiably composite finite).

Remark. In contrast with the DC-finite, another approach to generalizing the holonomic sequences is through the D-finite generating function (of a holonomic sequence). This was considered in [6] and the resulting generating function is known as *DD-finite*.

Proof. Following the idea of the proof of Theorem 12, assume that a_n satisfies the relation

$$C_{r,n} a_{n+r} + C_{r-1,n} a_{n+r-1} + \cdots + C_{0,n} a_n = 0,$$

where $C_{t,n}$ can be written in a closed-form as $C_{t,n} = \sum_{\alpha \in S_t} p_{t,\alpha}(n) \alpha^n$. We denote by $b_{s,t,\alpha}$ the coefficient of $n^s \alpha^n a_{n+t}$ in the relation above. Then, $p_{t,\alpha}(n) = \sum_{s=0}^{k} b_{s,t,\alpha} n^s$, and so the C^2-finite relation becomes

$$\sum_{t=0}^{r} \sum_{\alpha \in S_t} \sum_{s=0}^{k} b_{s,t,\alpha} n^s \alpha^n a_{n+t} = 0. \tag{18}$$

We next prove the following identity. For fixed s and t,

$$\begin{aligned}
\sum_{j=0}^{s} c_j^{(s,t)} x^{j-t} f^{(j)}(\alpha x) &= \sum_{j=0}^{s} c_j^{(s,t)} \sum_{n=0}^{\infty} (n)_j a_n \alpha^n x^{n-t} \\
&= \sum_{j=0}^{s} c_j^{(s,t)} \sum_{n=-t}^{\infty} (n+t)_j a_{n+t} \alpha^{n+t} x^n \\
&= \alpha^t \sum_{n=-t}^{\infty} \left[\sum_{j=0}^{s} c_j^{(s,t)} (n+t)_j \right] a_{n+t} (\alpha x)^n.
\end{aligned}$$

For each pair of (s,t), we solve for constants $c_j^{(s,t)}, \quad j = 0,1,2,\dots,s$, by equating coefficients of n^j in the equation $\sum_{j=0}^{s} c_j^{(s,t)}(n+t)_j = n^s$. Now, define $A_{s,t,\alpha}(x) = \sum_{n=0}^{\infty} a_{n+t} n^s (\alpha x)^n$ for fixed $s,t \geq 0$. Then,

$$\sum_{j=0}^{s} c_j^{(s,t)} x^{j-t} f^{(j)}(\alpha x) = \alpha^t \sum_{n=-t}^{\infty} a_{n+t} n^s (\alpha x)^n = \alpha^t A_{s,t,\alpha}(x) + \alpha^t \sum_{n=-t}^{-1} a_{n+t} n^s (\alpha x)^n. \tag{19}$$

From the C²-finite relation of a_n in (18), multiply x^n through, and sum n from 0 to ∞, we obtain

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}\sum_{t=0}^{r}\sum_{\alpha\in S_t}\sum_{s=0}^{k} b_{s,t,\alpha} n^s \alpha^n a_{n+t} x^n = \sum_{t=0}^{r}\sum_{s=0}^{k}\sum_{\alpha\in S_t} b_{s,t,\alpha} A_{s,t,\alpha}(x) \\ &= \sum_{t=0}^{r}\sum_{s=0}^{k}\sum_{\alpha\in S_t} b_{s,t,\alpha}\left[\alpha^{-t}\sum_{j=0}^{s} c_j^{(s,t)} x^{j-t} f^{(j)}(\alpha x) - \sum_{n=-t}^{-1} a_{n+t} n^s (\alpha x)^n\right], \quad \text{from (19).} \end{aligned}$$

Multiply x^r on both sides and rearrange this equation:

$$\sum_{t=0}^{r}\sum_{s=0}^{k}\sum_{\alpha\in S_t} b_{s,t,\alpha}\alpha^{-t}\sum_{j=0}^{s} c_j^{(s,t)} x^{j+r-t} f^{(j)}(\alpha x) = \sum_{t=0}^{r}\sum_{s=0}^{k}\sum_{\alpha\in S_t} b_{s,t,\alpha}\sum_{n=0}^{t-1} a_n (n-t)^s \alpha^{n-t} x^{n+r-t}.$$

We see that the left hand side is the differential equation of order at most k and degree at most $r+k$. The right hand side is the polynomial of degree at most $r-1$. □

Example 1: Let $a_{n+1} = F_{n+2} \cdot a_n$. Then,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} F_{n+1} \cdot a_{n-1} x^n = a_0 + x\sum_{n=1}^{\infty}(c_1\alpha_+^{n+1} + c_2\alpha_-^{n+1}) \cdot a_{n-1} x^{n-1} \\ &= a_0 + c_1\alpha_+^2 x \sum_{n=0}^{\infty} \alpha_+^n \cdot a_n x^n + c_2\alpha_-^2 x \sum_{n=0}^{\infty} \alpha_-^n \cdot a_n x^n, \quad \text{(shift index } n \text{ by 1)} \\ &= a_0 + c_1\alpha_+^2 x f(\alpha_+ x) + c_2\alpha_-^2 x f(\alpha_- x), \end{aligned}$$

where α_+ and α_- are the roots of equation $x^2 - x - 1 = 0$.

Maple Program

```
> C2ToDiff(N-(c1*a^(n+2)+c2*b^(n+2)),{1,a,b},
[a0],n,N,x,D);
```

We still consider a first-order relation in the next example, but this time the coefficient C_n has a polynomial factor.
Example 2: Let $a_{n+1} = (n+1)2^n \cdot a_n$. Then,

$$\begin{aligned} f(x) = \sum_{n=0}^{\infty} a_n x^n &= a_0 + \sum_{n=1}^{\infty} n2^{n-1} \cdot a_{n-1} x^n = a_0 + x \sum_{n=0}^{\infty} (n+1)2^n \cdot a_n x^n \\ &= a_0 + x^2 \sum_{n=1}^{\infty} n2^n \cdot a_n x^{n-1} + x \sum_{n=0}^{\infty} 2^n \cdot a_n x^n \\ &= a_0 + x^2 f'(2x) + x f(2x). \end{aligned}$$

Maple Program

```
> C2ToDiff(N-(n+1)*2^n,{1,2},[a0],n,N,x,D);
```

Let us now consider a second-order example.
Example 3: Let $a_{n+2} = a_{n+1} + 2^n \cdot a_n$. Then

$$\begin{aligned} f(x) = \sum_{n=0}^{\infty} a_n x^n &= a_0 + a_1 x + \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} 2^{n-2} \cdot a_{n-2} x^n \\ &= a_0 + a_1 x - a_0 x + x \sum_{n=0}^{\infty} a_n x^n + x^2 \sum_{n=0}^{\infty} a_n (2x)^n \\ &= a_0 + (a_1 - a_0)x + x f(x) + x^2 f(2x). \end{aligned}$$

Maple Program

```
> C2ToDiff(N^2-N-2^n,{1,2},[a0,a1],n,N,x,D);
```

Similar to holonomic sequences, the differential equation (17) can be made homogeneous by differentiating (17) once and combining the two equations. Hence, we obtain the following corollary.

Corollary 18. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where a_n is a C^2-finite sequence of order r and degree k. Then, $f(x)$ satisfies a homogeneous linear differential equation with polynomial coefficients*

$$\sum_{\alpha} \left[q_{\alpha,0}(x) f(\alpha x) + q_{\alpha,1}(x) f'(\alpha x) + \cdots + q_{\alpha,r'}(x) f^{(r')}(\alpha x) \right] = 0, \quad (20)$$

where the order r' is at most $k+1$, and the degree of $q_{\alpha,t}(x)$ for each α, t is at most $2r + k - 1$.

Example: The homogeneous differential equation of $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where $a_{n+1} = F_{n+2} \cdot a_n$ is

$$f'(x) - c_1 \alpha_+^2 [f(\alpha_+ x) + x f'(\alpha_+ x)] - c_2 \alpha_-^2 [f(\alpha_- x) + x f'(\alpha_- x)] = 0.$$

Maple Program

```
> C2ToDiffHom(N-(c1*a^(n+2)+c2*b^(n+2)),{1,a,b},
[a0],n,N,x,D);
```

The next theorem ensures that we can always find a C^2-finite recurrence relation for the coefficients a_n of $f(x)$ which satisfies a homogeneous linear differential equation of composite variables with polynomial coefficients.

Theorem 19. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Assume $f(x)$ satisfies a homogeneous linear differential equation with polynomial coefficients of order r and degree k*

$$\sum_{\alpha} \left[q_{\alpha,0}(x) f(\alpha x) + q_{\alpha,1}(x) f'(\alpha x) + \cdots + q_{\alpha,r}(x) f^{(r)}(\alpha x) \right] = 0. \quad (21)$$

Then, a_n is a C^2-finite sequence of order at most $r+k$ and degree at most r.

Proof. Assume that $q_{\alpha,t}(x) = \sum_{s=0}^{k} b_{\alpha,s,t} x^s$. Then, for each s, t, α,

$$x^s f^{(t)}(\alpha x) = \sum_{n=t}^{\infty} (n)_t a_n \alpha^n x^{n-t+s} = \alpha^{t-s} \sum_{n=s}^{\infty} (n+t-s)_t a_{n+t-s} (\alpha x)^n.$$

Now (21) becomes

$$\sum_{\alpha} \left[\sum_{t=0}^{r} \sum_{s=0}^{k} b_{\alpha,s,t} \alpha^{t-s} \sum_{n=s}^{\infty} (n+t-s)_t a_{n+t-s} (\alpha x)^n \right] = 0,$$

and so for each $n \geq 0$, a_n satisfies the recurrence

$$\sum_{t=0}^{r} \sum_{\alpha} \sum_{s=0}^{\min(n,k)} \left[b_{\alpha,s,t} (n+t-s)_t \alpha^{n+t-s} \right] a_{n+t-s} = 0.$$

Proposition 11 implies that the coefficient $C_{i,n} = \sum_{\alpha} \sum_{s=0}^{\min(n,k)} b_{\alpha,s,s+i} (n+i)_{s+i} \alpha^{n+i}$, for each i, is a C-finite sequence. It follows that a_n satisfying

$$C_{r,n} a_{n+r} + C_{r-1,n} a_{n+r-1} + \cdots + C_{-k,n} a_{n-k} = 0,$$

is a C^2-finite sequence. The claim of the order and degree is now immediate. □

5. Closure properties:

Theorem 20. *Assume a_n and b_n are C^2-finite sequences. The following are also C^2-finite sequences.*
(i) addition: $(a_n + b_n)_{n=0}^{\infty}$,

(ii) term-wise multiplication: $(a_n \cdot b_n)_{n=0}^{\infty}$,
(iii) Cauchy product: $(\sum_{i=0}^{n} a_i \cdot b_{n-i})_{n=0}^{\infty}$,
(iv) partial sum: $(\sum_{i=0}^{n} a_i)_{n=0}^{\infty}$,
(v) linear subsequence: $(a_{mn})_{n=0}^{\infty}$, *where* m *is a positive integer.*

The proof is along the same line as that of Theorem 15 for holonomic sequences, and we shall not repeat it here. This section will be devoted to a detailed discussion and examples instead.

Remark. The reader may have noticed that this time we have not specified the upper bound of the order in the theorem. While the same bounds as those used for holonomic sequences could be imposed, it is worth repeating here that for a C^2-finite sequence of order r, the leading coefficient $C_{r,n}$ must not be zero for any $n \geq 0$. This condition makes it not straightforward to determine a general bound for the order of the sequence. The first example below (from [5]) illustrates this issue.

Example 1: Let a_n and b_n be a C^2-finite sequence of order 1 defined by

$$\begin{aligned} a_{n+1} + (-1)^n a_n &= 0, \\ b_{n+1} + b_n &= 0. \end{aligned}$$

Let $c_n = a_n + b_n$ for $n \geq 0$. A recurrence of order 2 for c_n is in the form

$$[1-(-1)^n]c_{n+2} + 2c_{n+1} + [1+(-1)^n]c_n = 0.$$

This recurrence does not satisfy the definition of C^2-finite as the leading term, $C_{2,n} = 1-(-1)^n$, contains infinitely many zeros.
On the other hand, a recurrence of order 3 makes c_n a C^2-finite sequence:

$$c_{n+3} + \frac{1}{2}\left[1+(-1)^n\right]c_{n+2} + \frac{1}{2}\left[1-(-1)^n\right]c_n = 0.$$

The following example illustrates the idea behind the derived recurrence relations under the addition and term-wise multiplication operations.

Example 2: Let a_n be a sequence that satisfies the relation

$$a_{n+1} = F_{n+2} \cdot a_n, \qquad a_0 = 1,$$

where F_n is the Fibonacci sequence. Let b_n be a sequence that satisfies the relation

$$b_{n+2} = b_{n+1} + 2^n b_n, \quad b_0 = 1, b_1 = 1.$$

- addition: $c_n = a_n + b_n$.
  To solve for the recurrence relation of c_n, we write c_n, c_{n+1}, c_{n+2} and c_{n+3} as a linear combination of a_n, b_n and b_{n+1}. That is,

$$\begin{bmatrix} c_n \\ c_{n+1} \\ c_{n+2} \\ c_{n+3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ F_{n+2} & 0 & 1 \\ F_{n+3}F_{n+2} & 2^n & 1 \\ F_{n+4}F_{n+3}F_{n+2} & 2^n & 2^{n+1}+1 \end{bmatrix} \cdot \begin{bmatrix} a_n \\ b_n \\ b_{n+1} \end{bmatrix}.$$

The C-finite solution P (in the null space) of the C-finite sequence matrix M is

$$P = [2^n F_{n+2}(F_{n+4}F_{n+3} - F_{n+3} - 2^{n+1}), \ldots],$$

which gives rise to a C^2-finite relation of c_n, $n \geq 1$, of order 3

$$\begin{aligned}
&2^n F_{n+2}(F_{n+4}F_{n+3} - F_{n+3} - 2^{n+1})c_n \\
&+ \left[F_{n+4}F_{n+3}F_{n+2} + 2^{2n+1} - 2^{n+1}F_{n+3}F_{n+2} - F_{n+3}F_{n+2}\right] c_{n+1} \\
&+ \left[2^{n+1}F_{n+2} + F_{n+2} - F_{n+4}F_{n+3}F_{n+2} + 2^n\right] c_{n+2} \\
&+ \left[F_{n+3}F_{n+2} - F_{n+2} - 2^n\right] c_{n+3} = 0.
\end{aligned}$$

- term-wise multiplication: $d_n = a_n \cdot b_n$.
  To solve for the recurrence relation of d_n, we write d_n, d_{n+1} and d_{n+2} as a linear relation of $a_n b_n$ and $a_n b_{n+1}$. That is,

$$\begin{bmatrix} c_n \\ c_{n+1} \\ c_{n+2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & F_{n+2} \\ 2^n F_{n+3}F_{n+2} & F_{n+3}F_{n+2} \end{bmatrix} \cdot \begin{bmatrix} a_n b_n \\ a_n b_{n+1} \end{bmatrix}.$$

  The C-finite solution P (in the null space) of the C-finite sequence matrix M is

$$P = [-2^n F_{n+2}F_{n+3}, -F_{n+3}, 1],$$

  yielding a C^2-finite relation of c_n, $n \geq 0$, of order 2:

$$-2^n F_{n+2}F_{n+3}c_n - F_{n+3}c_{n+1} + c_{n+2} = 0.$$

As we mentioned earlier, the proof of closure properties for C^2-finite is similar to the holonomic one. It turns out that we can directly use the same Maple code to get recurrence relations for C^2-finite:

Maple Program

```
> HoAdd(N-F(n+2),N^2-N-2^n,n,N,c);
> HoTermWise(N-F(n+2),N^2-N-2^n,n,N,c);
```

6. Asymptotic approximation solutions:
We have already seen that the rate of growth of C-finite is $\mathcal{O}(\alpha^n)$ for some constant α. Here, the rate of growth of C^2-finite is $\mathcal{O}(\alpha^{n^2})$ for some constant α. Also, we have presented a procedure to obtain an asymptotic approximation solution for the holonomic recurrence in the previous section. It appears, however, to be more difficult to derive asymptotic approximation solutions for the C^2-finite recurrences, and merits further investigation. We leave this as an open problem.
Open problem 3: Find asymptotic approximations of solutions to the C^2-finite sequences.

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