On Arithmetic Combinatorics

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Introduction

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Ramsey theory is Paul Erdős first love and remains his love through out his career. It is also the central interest of the branch of mathematic called *Hungarian mathematics*.

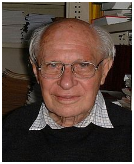
Erdős-Szekeres theorem

In 1935 Erdős and Szekeres showed the following theorem:

Theorem (Monotone Subsequence theorem)

Any sequence of $n^2 + 1$ integers contains a monotonic subsequence of length $n + 1$.

Paul Erdős



George Szekeres

Van der Waerden Numbers

Theorem (Van der Waerden's Theorem, 1927)

For all positive integers k and r, there exists a least positive integer $w(k,r)$ such that for every r-coloring of $[1, w(k,r)]$, there is a monochromatic arithmetic progression of length k.

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$k \backslash r$	2	3	4	5
3	9	27	76	≥ 171
4	35	293	≥ 1049	≥ 2255
5	178	≥ 2174	≥ 17706	≥ 98749
6	1132	≥ 11192	≥ 91332	≥ 540026
7	≥ 3704	≥ 48812	≥ 420218	≥ 1381688

Lower bounds and values of $w(k,r)$

$w(2,r) = r + 1$, why?

An Upper Bound of Van der Waerden Numbers

It is very difficult to determine good upper bounds for $w(k, r)$ (although they exist by Van der Waerden's theorem).

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Definitions

Let $f_1(k) = 2k, k \geq 1$

$f_2(k) = f_1^{(k)}(1) = 2^k$

$f_3(k) = f_2^{(k)}(1) = 2^{2^{2^{\cdots}}}$, k times. We call $f_3(k)$ a tower function.

$f_4(k) = f_3^{(k)}(1)$. We define $wow(k) = f_4(k)$.

Then we define Ackermann function, $ack(k)$ by

$$ack(k) = f_k(k).$$

An Upper Bound of Van der Waerden Numbers

The original proof of Van der Waerden's theorem gives us an phenomenal upper bounds. It can be shown that

$$w(k, 2) \leq ack(k), \;\; 10 \leq k.$$

In 1987, Saharon Shelah used an argument that is fundamentally different than the original to prove that

$$w(k, 2) \leq wow(k).$$

In 1997-1998, Timothy Gowers use Fourier analysis to improve, among other results of this type, this upper bound.

$$w(k, 2) \leq 2^{2^{2^{2^{2^{k+9}}}}}.$$

This bound is much smaller than the tower function.

Hales-Jewett Theorem

"Hales-Jewett theorem strips van der Waerden's theorem of its unessential elements and reveals the heart of Ramsey theory." – Graham, Rothschild, Spencer, *Ramsey Theory*

Theorem (Hales-Jewett)

For all k, r there exist the least positive integer, $H(k, r)$, so that, for $n \geq H(k, r)$, if the n-dimensional cube

$$\{(x_1, x_2, ..., x_n) : x_i \in \{1, 2, ..., k\}\}$$

is r-colored, there exists a monochromatic "line".

Hales-Jewett Theorem

The only known Hales-Jewett number is $H(3, 2) = 4$ by Neil Hindman and Eric Tressler (2014).

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For more tic-tac-toe type problems and variations see the book:
Combinatorial Games (2008) by József Beck.

Density Version

Conjecture (Density Version; Posted by Erdős, 1936)

Let A be a subset of $\mathbb{Z}$ such that

$$\limsup_{n\to\infty} \frac{|A \cap \{0, \ldots, n-1\}|}{n} > 0.$$

Then A contains arbitrarily long arithmetic progressions .

Density Version

The first proof due to Roth (1952) was on $k = 3$ and used Fourier analysis instead of combinatorics. Later in 1974, Szemeredi solved the general conjecture by means of sophisticated combinatorics. In the process he create the so called "Szemerdi regularity lemma". In 1976 Furstenberg reproved it by using ergodic theory. Finally in 1998 Gowers proved it again by using harmonic analysis. This is part of the work mentioned earlier.

Roth's Theorem

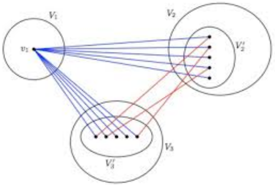
Theorem (Roth's Theorem)

Let $\delta > 0$ be given, and let $A \subset \{1, 2, \ldots, N\}$ be set of size $|A| = \delta N$. If $N > e^{e^{\frac{C}{\delta}}}$ for some absolute constant C then A contains an arithmetic progression of length 3.

Szemerdi Regularity Lemma

The Szemerédi regularity lemma states that every large enough graph can be divided into subsets of about the same size so that the edges between different subsets behave almost randomly.

Endre Szemerédi



"Regular" Graph

Erdős Conjecture II

In a 1976 talk titled "To the memory of my lifelong friend and collaborator Paul Turán" Erdős proposed the stronger conjecture.

Conjecture (Erdős, 1976)

If

$$\sum_{n \in A} \frac{1}{n} = \infty$$

then A contains arithmetic progressions of any given length.

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Conjecture (Erdős, 1976)

If

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then A contains arithmetic progressions of any given length.

The conjecture is still open and worth \$5000.
However, in 2004, Ben Green and Terence Tao proved the special case of this conjecture when A is the set of prime numbers.

Finite Analogs of Szemerédi's Theorem

Material on this part is taken from Paul Raff and Doron Zeilberger (2010).

Theorem (Szemerédi's Theorem)

Given an integer $n \geq 1$ and an integer $k \geq 3$, let $r_k(n)$ denote the size of any largest subset S of $\{1, 2, \ldots, n\}$ for which there are **no** *subsets of the form*

$$\{i, i+d, i+2d, \ldots, i+(k-1)d\} \quad (i \geq 1 \quad , \quad 1 \leq d < \infty) \quad ,$$

then $r_k(n) = o(n)$.

Finite Analogs of Szemerédi's Theorem

Theorem (Finite Version of Szemerédi's Theorem)

Given an integer $n \geq 1$ and integers $k \geq 3$, $D \geq 1$, let $R_{k,D}(n)$ denote the size of any largest subset S of $\{1, 2, \ldots, n\}$ for which there are **no** *subsets of the form*

$$\{i, i+d, i+2d, \ldots, i+(k-1)d\} \quad (\ i \geq 1 \quad , \quad 1 \leq d \leq D\ ) \quad ,$$

then there exists a rational number $\alpha_{k,D}$ such that

$$\lim_{n \to \infty} \frac{R_{k,D}(n)}{n} = \alpha_{k,D} \quad .$$

(Rigorous!) Values for $\alpha_{k,D}$

$D/\ k$	3	4	5	6	7	8	9	10	11	12	13	14
1	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{10}{11}$	$\frac{11}{12}$	$\frac{12}{13}$	$\frac{13}{14}$
2	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{8}{9}$	$\frac{8}{9}$	$\frac{10}{11}$	$\frac{10}{11}$	$\frac{12}{13}$	$\frac{12}{13}$
3	$\frac{4}{8}$	$\frac{8}{12}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{20}{23}$	$\frac{10}{11}$	$\frac{10}{11}$	$\frac{12}{13}$	$\frac{12}{13}$
4	$\frac{4}{9}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{26}{30}$	$\frac{10}{11}$	$\frac{10}{11}$	$\frac{12}{13}$	$\frac{12}{13}$
5	$\frac{4}{9}$	$\frac{4}{7}$	$\frac{16}{24}$	$\frac{22}{30}$	$\frac{6}{7}$							
6	$\frac{4}{9}$	$\frac{4}{7}$										
7	$\frac{4}{9}$	$\frac{6}{11}$										
8	$\frac{4}{9}$	$\frac{6}{11}$										
9	$\frac{4}{10}$											
10	$\frac{4}{11}$											

Values of $\alpha_{k,D}$

Asymptotic Minimum Triples

This is the density version of my favorite Ramsey problem, asymptotic minimum triples.

Problem

Given a large number N and the density α, $0 \le \alpha \le 1$.
Find the minimum number of 3-progressions on $A \subset \{1, 2, \ldots, N\}$ with $|A| = \alpha N$.

Thank you for Your Attention!

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Any Questions?