Average Length of the Games

Thotsaporn "Aek" Thanatipanonda

Mahidol University International College

June 5-7, 2019

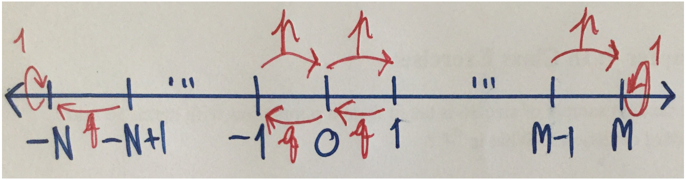
Method of Guess and Check

Many problems in combinatorics can be naturally stated in term of recurrence relation. We consider various ansatz, i.e. poynomial ansatz, C-finite ansataz or holonomic ansatz for the recurrence. Once the answer has been guessed, we can formally verified it by plugging it back to the recurrence and checking the initial condition.

I will demonstrate this method on the problems related to games.

Gambler's Ruin

The game starts with the gambler has 0 chip. For each play, the probability to gain one chip is $p = 1/2$ and the probability to lose one chip is $q = 1 - p = 1/2$. The game will continue until the gambler gains M chips or loses N chips. Find the expected number of plays until the game ends. Let $T(a)$ be the expected number of play until the game ends starting at a chips.



Gambler's Ruin

Recurrences:

$$T(M) = T(-N) = 0,$$

$$T(a) = 1 + \frac{1}{2}T(a+1) + \frac{1}{2}T(a-1) \quad \text{for} -N < a < M.$$

Example: For $N = 6, M = 5$, $T(a)$ for $a = -6, \ldots, 5$ are

$$0, 10, 18, 24, 28, 30, 30, 28, 24, 18, 10, 0.$$

Gambler's Ruin

Recurrences:

$$T(M) = T(-N) = 0,$$

$$T(a) = 1 + \frac{1}{2}T(a+1) + \frac{1}{2}T(a-1) \quad \textit{for } -N < a < M.$$

Example: For $N = 6, M = 5, \quad T(a)$ for $a = -6, \ldots, 5$ are

$$0, 10, 18, 24, 28, 30, 30, 28, 24, 18, 10, 0.$$

The solution to this system is

$$T(a) = (N + a) \cdot (M - a), \quad -N \leq a \leq M.$$

Gambler's Ruin

We can explore more with different gains and losses. For example, if we gain 2 chips with $p = 1/2$ and lose one chip with $q = 1 - p = 1/2$. Then the recurrences:

$$T_2(M+1) = T_2(M) = T_2(-N) = 0,$$

$$T_2(a) = 1 + \frac{1}{2}T_2(a+2) + \frac{1}{2}T_2(a-1) \quad \textit{for} - N < a < M.$$

Example: For $N = 6, M = 5, \quad T(a)$ for $a = -6, \ldots, 5$ are

$$0, \frac{195}{29}, \frac{294}{29}, \frac{332}{29}, \frac{335}{29}, \frac{312}{29}, \frac{280}{29}, \frac{231}{29}, \frac{190}{29}, \frac{124}{29}, \frac{91}{29}, 0.$$

Gambler's Ruin

The recurrence for any $N \geq 0$ and $M \geq 0$ is

$$T_2(a+4) = T_2(a+3) + 2T_2(a+2) - 3T_2(a+1) + T_2(a).$$

In the operator notation i,e. $A \cdot T_2(a) = T_2(a+1)$.

$$(A^4 - A^3 - 2A^2 + 3A - 1) \cdot T_2(a) = (A-1)^2(A^2 + A - 1) \cdot T_2(a) = 0.$$

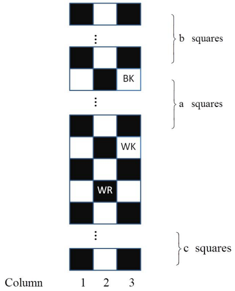
The solution of $T_2(a)$ is in the form

$$T_2(a) = c_0 + c_1 a + c_2\alpha_1^a + c_3\alpha_2^a,$$

where α_1 and α_2 are the roots of $N^2 + N - 1 = 0$ i.e. $\alpha_1 = \dfrac{-1+\sqrt{5}}{2}$ and $\alpha_2 = \dfrac{-1-\sqrt{5}}{2}$

Rook Endgame Problems

We consider Chess played on an $3 \times n$ board, with only the two kings and the white rook remaining, but placed at arbitrary positions. We prove that for almost all initial positions, White can checkmate Black in $\leq n + 2$ moves, and that this upper bound is sharp.



Standard Position on a $3 \times n$ Board

Rook Endgame Problems

Let x be the column of the black King.
Let y be the column of the white King.
Let z be the column of the white Rook.
Let $f_{x,y,z}(a,b)$ be the number of moves needed to checkmate with the above initial position.

$$
\begin{aligned}
f_{1,1,1}(a,b) &\leq a+b+1 \quad , a \geq 2. \\
f_{1,1,2}(a,b) &\leq \begin{cases} a+b+1 & , a \text{ is even and } a \geq 2. \\ a+b & , a \text{ is odd and } a \geq 2. \end{cases} \\
f_{1,1,3}(a,b) &\leq a+b+1 \quad , a \geq 2. \\
&\ldots
\end{aligned}
$$

Rook Endgame Problems

Proof.

Base Case: Verify that all the conjectures are true for $a + b \leq 3$ where $a \geq 0, b \geq 0$.

Induction Step: Consider $f_{1,1,1}(a, b)$;

Case 1: a is even.

White chooses to move his Rook to the second column. Black's only legal moves are to move his King up or down the first column. By the inductive hypothesis, in case the black King moves up, it would take at most $f_{1,1,2}(a+1, b-1) = (a+1) + (b-1) = a + b$ moves to checkmate. In case the black King moves down, it would take at most $f_{1,1,2}(a-1, b+1) = (a-1) + (b+1) = a + b$ moves to checkmate. Therefore case 1 takes at most $(a+b)+1$ moves to checkmate.

. . .

□

Pile Game

The game starts with 0 chip in the pile. The player randomly adds a or b chips, $0 \leq a \leq b$, with probability $p = 1/2$ each. Find the average number of turns to reach a pile of n chips.
Let $T(k)$ be the expected number of play until reaching n chips starting at k chips.

Recurrences:

$$T(n+i) = 0, \quad 0 \leq i \leq b.$$

$$T(k) = 1 + \frac{1}{2}T(k+a) + \frac{1}{2}T(k+b), \quad \textit{for} \;\; 0 \leq k < n.$$

Pile Game

This opens the door for an exploration. Let's redefine function $T(k)$ in a more standard way. Let $S(n)$ be the average number of turns to reach the pile of n chips starting from an empty pile.

With the choice of removing $a = 1$ and $b = 2$ chips, we have that

$$S(n) = \frac{2n}{3} + \frac{2}{9} - \frac{2}{9}\left(-\frac{1}{2}\right)^n.$$

Another example, with the choice of removing $a = 2$ and $b = 3$ chips,

$$S(n) = \frac{2n}{5} + \frac{8}{25} - \frac{(4+3i)}{25}\left(\frac{-1+i}{2}\right)^n - \frac{(4-3i)}{25}\left(\frac{-1-i}{2}\right)^n.$$

Pile Game

Remark:
Another way is to use definition of expectation. Let X be a random number of turns until the game ends.

$$E[X] = \sum x \cdot p(x).$$

Pile Game: $a = -1, b = 1$

Next we investigate the case $a < 0$ and $b > 0$: In this case we cannot write the recurrence anymore since the function can go unbounded.

Example 1:
Let $a = -1, b = 1$. We consider $S(n)$, the average number of turns to reach n chips, $n > 0$.

Let first try to find $S(1)$. Let p_k be the probability that the games ended at move k, i.e. $p_1 = \frac{1}{2}$. The sequence of $2^n \cdot p_k$ looks like

$$1, 0, 1, 0, 2, 0, 5, 0, 14, 0, 42, 0, 132, 0, 429, 0, 1430, \ldots$$

Pile Game: $a = -1, b = 1$

Therefore we see that $p_{2k} = 0$ and $p_{2k+1} = \left(\frac{1}{2}\right)^{2k+1} \frac{\binom{2k}{k}}{k+1}$. Then

$$S(1) = \sum_{k=0}^{\infty} (2k+1) \left(\frac{1}{2}\right)^{2k+1} \frac{\binom{2k}{k}}{k+1}.$$

After typing this sum in maple, it returns ∞. So we conclude that $S(n) = \infty$ for all $n > 0$.

Pile Game: $a = -1, b = 2$

Let $a = -1, b = 2$. We consider $S_2(n)$, the average number of turns to reach n chips, $n > 0$.

Let first try to find $S_2(1)$. Let p_k be the probability that the games ended at move k. For example $p_1 = \frac{1}{2}$. The sequence of $2^k \cdot p_k$ looks like

$$1, 1, 0, 1, 2, 0, 3, 7, 0, 12, 30, 0, 55, 143, 0, 273, 728 \ldots$$

Pile Game: $a = -1, b = 2$

We see that $p_{3k} = 0$, $p_{3k+1} = \left(\frac{1}{2}\right)^{3k+1} \frac{\binom{3k}{k}}{2k+1}$ and $p_{3k+2} = \left(\frac{1}{2}\right)^{3k+2} \frac{\binom{3k+1}{k}}{k+1}$. Then

$$S_2(1) = \sum_{k=0}^{\infty} \left[(3k+1) \left(\frac{1}{2}\right)^{3k+1} \frac{\binom{3k}{k}}{2k+1} + (3k+2) \left(\frac{1}{2}\right)^{3k+2} \frac{\binom{3k+1}{k}}{k+1} \right].$$

After typing this sum in maple, it returns

$$hypergeom\left(\left[\frac{2}{3}, 1, \frac{4}{3}, \frac{9}{5}\right], \left[\frac{4}{5}, \frac{3}{2}, 2\right], \frac{27}{32} \right).$$

Snakes and Ladders Board Game

You roll a six sided die and move accordingly with starting at $a = 0$ and the game ends if you reach $a = n$ or more. If you land on the bottom of the ladder you move to the top of the ladder. If you land on the top of the snake you move to the tail of the snake.

No Ladder, No Snake Case

Let $T(n)$ be the average number of turns until the game ends on the board of size n. (always starts at $a = 0$).

Recurrences:

$$T(-n) = 0, \qquad n = 0, 1, 2, 3, 4, 5.$$

$$T(n) = 1 + \frac{1}{6}[T(n-1) + T(n-2) + T(n-3) + T(n-4) + T(n-5) + T(n-6)]$$

for $\quad n \geq 1.$

No Ladder, No Snake Case

The corresponding recurrence (C-finte) from maple is

$$(6N^5 + 5N^4 + 4N^3 + 3N^2 + 2N + 1)(N - 1)^2 = 0.$$

We find $T(n)$ algebraically from this recurrence (along with initial conditions) or combinatorially from the definition of $E[X]$.

Proposition

$$T(n) = \frac{2n}{7} + \frac{10}{21} + O(\alpha^n), \quad \textit{for some } |\alpha| < 1.$$

One Snake at the End and Go Back to the Starting Point

Consider again rolling a six sided die and move accordingly with starting at $a = 0$ and the game ends at $a \geq n + 1$. If you land at $a = n$ then you move back to $a = 0$. Let $S_n(a)$ be the average number of play starting at a until the game ends, i.e. $S_n(a) = 0, \quad for \quad a \geq n + 1$.

One Snake at the End and Go Back to the Starting Point

We first note about, $p(n)$, the probability that starting at $a = 0$ and land on $a = n$. Recurrence of $p(n)$:

$$p(n) = \frac{1}{6}[p(n-1) + p(n-2) + p(n-3) + p(n-4) + p(n-5) + p(n-6)].$$

The sequence $p(n), \quad n \geq 0$ looks like

$$1, \frac{1}{6}, \frac{7}{36}, \frac{49}{216}, \frac{343}{1296}, \frac{2401}{7776}, \frac{16807}{46656}, \frac{70993}{279936}, \ldots$$

One Snake at the End and Go Back to the Starting Point

The sequence satisfies the similar looking recurrence of $T(n)$.

$$\begin{aligned} 0 &= 6N^6 - N^5 - N^4 - N^3 - N^2 - N - 1 \\ &= (N-1)(6N^5 + 5N^4 + 4N^3 + 3N^2 + 2N + 1). \end{aligned}$$

The formula for $p(n)$ can also be derived algebraically or combinatorially.

Lemma

$$p_n = \frac{2}{7} + O(\alpha^n), \quad \textit{for some } |\alpha| < 1.$$

One Snake at the End and Go Back to the Starting Point

Now let's get back to $S_n(a)$:

Recurrences:

$$\begin{aligned} S_n(n+i) &= 0, \qquad i = 1,2,3,4,5, \\ S_n(n) &= S_n(0), \\ S_n(a) &= 1 + \frac{1}{6}[S_n(a+1) + S_n(a+2) + S_n(a+3) \\ &\quad + S_n(a+4) + S_n(a+5) + S_n(a+6)] \\ &\quad \text{for} \quad 0 \le a < n. \end{aligned}$$

One Snake at the End and Go Back to the Starting Point

Proposition

$$\lim_{n\to\infty} \frac{S_n(0)}{T(n)} = \frac{7}{5}.$$

One Snake at the End and Go Back to the Starting Point

Corollary

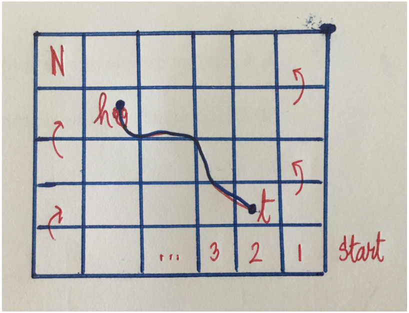
$$S_n(0) = \left(\frac{7}{5} + O(\alpha_1^n)\right) T(n) = \frac{2n}{5} + \frac{2}{3} + O(\alpha_2^n),$$

for some $|\alpha_1| < 1, |\alpha_2| < 1.$

We could be sure that the formula is accurate by comparing the formula with the exact values from the recurrences. Moreover we can do simulation for number of turns and apply the Central Limit Theorem to the confidence interval.

The General Form of One Snake Case

Similar to the previous case, however this time The head of the snake is at position $a = h$ and the tail is at position $a = t$ with the board of size n. Let $R_n(a)$ be the average number of play starting at a until the game ends.



The General Form of One Snake Case

Recurrences:

$$\begin{aligned}
R_n(n+i) &= 0, \qquad i = 0,1,2,3,4,5, \\
R_n(h) &= R_n(t), \\
R_n(a) &= 1 + \frac{1}{6}[R_n(a+1) + R_n(a+2) + R_n(a+3) \\
&\quad + R_n(a+4) + R_n(a+5) + R_n(a+6)] \\
&\quad \text{for} \quad a = \{0,1,2,\dots,h-1,h+1,\dots,n-1\}.
\end{aligned}$$

The General Form of One Snake Case

The sequence $R_n(0), \quad n \geq 0$ is probably not a C-finite. But we can still find a formula based on the idea previously.

Proposition

For some large $h - t$ and $n - h$,

$$R_n(0) = \frac{2n}{7} + \frac{10}{21} + \frac{4(h-t)}{35}.$$

The other more complicated scenarios are still left open.

Simplified Dreidel

The motivation for this whole average number of turns is the game called Dreidel. Here we only consider the simplified version:

Consider the two players with pot game where both players started with a and b amount of chips respectively. First player flips a fair coin to decide whether to pay 1 chip to the pot or take the whole pot and then each player donates 1 chip to the pot. The game ends if one of the player has no chip in their possession.

Let $T(a, p, b)$ be the average number of plays until the game ends where first player has a chips, second player has b chips and there are p chips in the pot.

Simplified Dreidel

Recurrences:

$$T(a,p,b) = 1 + \frac{1}{2}\left[T(b,p+1,a-1) + T(b-1,2,a+p-1)\right],$$

$$T(0,p,b) = T(a,p,0) = 0, \qquad \text{for any } a \geq 0, p \geq 0, b \geq 0.$$

Example: Some values of $T(a,p,b)$ when $a+b+p=6$ are

$$T(1,1,4) = \frac{33}{16}, \quad T(1,2,3) = \frac{5}{2}, \quad T(1,3,2) = \frac{9}{4}, \quad T(1,4,1) = 1,$$

$$T(2,1,3) = \frac{57}{16}, \quad T(2,2,2) = 3, \quad T(2,3,1) = \frac{3}{2},$$

$$T(3,1,2) = \frac{15}{4}, \quad T(3,2,1) = \frac{17}{8}, \quad T(4,1,1) = \frac{9}{4}.$$

Open Problem

Open Problem:
It would be too much to ask for the exact formula for $T(a, p, b)$. But, instead, find a reasonable quadratic, in a, p, b, upper bound (or lower bound) for $T(a, p, b)$.

References

- H. H. Leung, T. Thanatipanonda, *A Probabilistic Two-Pile Game*, to appear in Journal of Integer Sequences.
- T. Robinson, S. Vijay, *Dreidel Lasts $O(N^2)$ Spins*, Advances in Applied Mathematics 36, 85-94, 2006.
- T. Thanatipanonda, *Rook Endgame Problems in m by n Chess*, Advances in Applied Mathematics, Volume 61, Page 19-24, October 2014.
- D. Zeilberger, *Does Dreidel Last O(NUTS**2) Spins?*, The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger, Kislev 30, 5760.
- D. Zeilberger, *The C-finite Ansatz*, Ramanujan Journal 31(2013), 23-32. [special issue in honor of Mourad Ismail and Dennis Stanton]