Beyond Zudilin's Conjectured q-analog of Schmidt's problem

Thotsaporn "Aek" Thanatipanonda
thotsaporn@gmail.com

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Abstract

Using the methodology of (rigorous) *experimental mathematics*, we give a simple and motivated solution to Zudilin's question concerning a q-analog of a problem posed by Asmus Schmidt about a certain binomial coefficients sum. Our method is based on two simple identities that can be automatically proved using the Zeilberger and q-Zeilberger algorithms. We further illustrate our method by proving two further binomial coefficients sums.

1 Introduction

In 1992, Asmus Schimdt [3], conjectured that for any integer $r \geq 1$, the sequence of numbers $\{c_k^{(r)}\}_{k\geq 0}$ defined implicitly by

$$\sum_k \binom{n}{k}^r \binom{n+k}{k}^r = \sum_k \binom{n}{k}\binom{n+k}{k} c_k^{(r)}, \quad n = 0, 1, 2, \ldots$$

are always integers. It took more than 10 years to completely solve this conjecture([5]). Shorter proofs were found recently ([1, 4]). The key step in [4] was the fact that $a_{k,j}^{(r)}$, defined implicitly by

$$\binom{n}{k}^r \binom{n+k}{k}^r = \sum_j a_{k,j}^{(r)} \binom{n}{j}\binom{n+j}{j}, \quad n = 0, 1, 2, \ldots \qquad (1)$$

are all integers.

The integral property of the $a_{k,j}^{(r)}$, in fact, directly solves the conjecture. It is very possible that had Schimdt conjectured this equation in the first place, his conjecture would have been proved much sooner.

Inspired by (1), we investigate binomial coefficients summands of the form

$$\prod_i \binom{n+d_i k}{b_i k + c_i},$$

where d_i, b_i, c_i are fixed integers, such that their powers can be written as an integer linear combination of themselves.

2 Results

In this section, we search for $f(n,k)$ for which $a_{k,j}^{(r)}$

$$f(n,k)^r = \sum_j a_{k,j}^{(r)} f(n,j), \quad n = 0, 1, 2, \ldots \tag{2}$$

are all integers.

It was not an accident that the $a_{k,j}^{(r)}$ happen to be integers. Experiments show that there exists an integer-valued function $S_f(k,j,i)$, free of r, such that $\bar{a}_{k,j}^{(r)}$ define by $\bar{a}_{k,k}^{(1)} = 1$, $\bar{a}_{k,j}^{(1)} = 0$ for $j \neq k$, and

$$\bar{a}_{k,j}^{(r+1)} = \sum_i S_f(k,j,i) \bar{a}_{k,i}^{(r)}. \tag{3}$$

agrees with $a_{k,j}^{(r)}$ in (2).

We prove this fact by showing that $\bar{a}_{k,j}^{(r)}$ also satisfy (2). The proof relies on an induction. On the one hand,

$$\begin{aligned}\sum_j \bar{a}_{k,j}^{(r+1)} f(n,j) &= \sum_j \sum_i S_f(k,j,i) \bar{a}_{k,i}^{(r)} f(n,j) \quad \text{(by definition of } \bar{a}_{k,j}^{(r+1)}\text{)} \\ &= \sum_i \bar{a}_{k,i}^{(r)} \sum_j S_f(k,j,i) f(n,j).\end{aligned}$$

On the other hand,

$$\begin{aligned} f(n,k)^{r+1} &= f(n,k)^r f(n,k) \\ &= \sum_i \bar{a}^{(r)}_{k,i} f(n,i) f(n,k) \quad \text{(by induction hypothesis).} \end{aligned}$$

Hence the proof boils down to the condition that $S_f(k,j,i)$ must satisfy:

$$f(n,i)f(n,k) = \sum_j S_f(k,j,i) f(n,j), \quad n = 0,1,2,3,.... \tag{4}$$

Let's state this observation as a theorem.

Theorem 2.1. *Given a pair $f(n,k)$ and $S(k,j,i)$ such that*

$$f(n,i)f(n,k) = \sum_j S(k,j,i) f(n,j), \quad \textit{for all } \; n,i,k \geq 0.$$

Define $a^{(r)}_{k,j}$ recursively by $a^{(1)}_{k,k} = 1$, $a^{(1)}_{k,j} = 0$ for $j \neq k$ and

$$a^{(r+1)}_{k,j} = \sum_i S(k,j,i) a^{(r)}_{k,i}.$$

Then for $k \geq 0$ and $r \geq 1$,

$$f(n,k)^r = \sum_j a^{(r)}_{k,j} f(n,j), \quad n = 0,1,2,....$$

Once we know where to look, of course in this case guided by Schidmt's problem, it becomes a routine job that computers are so good at. We first pick some binomial term $f(n,k)$. Then we crank out some numerical values of $S(k,j,i)$ according to (4). Once we have enough data, we ask our computer to guess the relation, or even the formula for $S(k,j,i)$. Finally, needless to say, the proof of the identities can be routinely done by Zeilberger's algorithm. Here is the list that we were able to find.

Result 2.1.1:
These are the functions we used in Schmidt's conjecture.

$$f(n,k) = \binom{n}{k}\binom{n+k}{k}, \quad S(k,j,i) = \binom{i+k}{i}\binom{j}{j-i,\, j-k,\, i+k-j}.$$

Result 2.1.2:
For any fixed integer c,

$$f(n,k)=\binom{n}{k+c}\binom{n+k}{k+c},\quad S(k,j,i)=\frac{(i+k+c)!}{(i+c)!(k+c)!}\frac{(j+c)!}{j!}\binom{j+c}{j-i,j-k,i+k+c-j}.$$

Result 2.2:

$$f(n,k)=\binom{n}{k},\quad S(k,j,i)=\binom{j}{j-i,j-k,i+k-j}.$$

Result 2.3:

$$f(n,k)=\binom{n+k}{k},\quad S(k,j,i)=(-1)^{i+j+k}\binom{j}{j-i,j-k,i+k-j}.$$

3 q-analog

We now present q-analogs of the results from section 2.

The q-binomial $\begin{bmatrix} n \\ k \end{bmatrix}$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}=\begin{cases} \frac{(q)_n}{(q)_k(q)_{n-k}} & \text{if } 0\le k\le n. \\ 0 & \text{otherwise} \end{cases}$$

where $(q)_0=1$ and $(q)_n=(1-q)(1-q^2)...(1-q^n)$ for $n=1,2,....$

The proof of Theorem 3.1 below is similar to the proof of Theorem 2.1. We leave the details to the reader.

Theorem 3.1. *Given a triple $f(n,k)$, $A(k,j,i,n)$ and $S(k,j,i)$ satisfying*

$$f(n,i)f(n,k)=\sum_j q^A S(k,j,i)f(n,j),\quad \textit{for all}\ \ n,i,k\ge 0. \tag{5}$$

Let $B(k,j,i)$ and $C(k,j,r,n)$ be any functions such that

$B(k,j,i)+C(k,j,r+1,n)=A(k,j,i,n)+C(k,i,r,n)$ *and*

$C(k,k,1,n) = 0.$

Define $P^{(r)}_{k,j}(q)$ *recursively by* $P^{(1)}_{k,k}(q) = 1,\ P^{(1)}_{k,j}(q) = 0$ *for* $j \neq k$ *and*

$$P^{(r+1)}_{k,j}(q) = \sum_i q^B S(k,j,i) P^{(r)}_{k,i}(q).$$

Then for $k \geq 0$ *and* $r \geq 1$*,*

$$f(n,k)^r = \sum_j q^C f(n,j) P^{(r)}_{k,j}(q), \quad n = 0, 1, 2, \ldots.$$

The proofs of (5) of the results below can again be done automatically using the q-Zeilberger algorithm. Once we find function A from (5), it is only a matter of simple calculations to solve for functions B and C.

Result 3.1.1:

$$f(n,k) = \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix}, \quad S(k,j,i) = \begin{bmatrix} i+k \\ i \end{bmatrix} \begin{bmatrix} j \\ j-i, j-k, i+k-j \end{bmatrix},$$

$$\begin{aligned} A(k,j,i,n) &= (n-j)(k+i-j), \\ B(k,j,i) &= -(k+i-j)j, \\ C(k,j,r,n) &= (rk-j)n. \end{aligned}$$

Result 3.1.2:

$$f(n,k) = \begin{bmatrix} n \\ k+c \end{bmatrix} \begin{bmatrix} n+k \\ k+c \end{bmatrix}, \quad S(k,j,i) = \frac{(q)_{i+k+c}}{(q)_{i+c}(q)_{k+c}} \frac{(q)_{j+c}}{(q)_j} \begin{bmatrix} j \\ j-i, j-k, i+k-j \end{bmatrix},$$

$$\begin{aligned} A(k,j,i,n) &= (n-j)(k+i-j) + c(n-k-i-c), \\ B(k,j,i) &= -(k+i-j)j - c(k+i+c), \\ C(k,j,r,n) &= (rk-j)n + rcn - cn. \end{aligned}$$

Result 3.2:

$$f(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}, \quad S(k,j,i) = \begin{bmatrix} j \\ j-i, j-k, i+k-j \end{bmatrix},$$

$$\begin{aligned} A(k,j,i,n) &= (j-i)(j-k), \\ B(k,j,i) &= (j-i)(j-k), \end{aligned}$$

$C(k, j, r, n) = 0.$

Result 3.3:

$$f(n,k) = \begin{bmatrix} n+k \\ k \end{bmatrix}, \quad S(k,j,i) = (-1)^{i+j+k} \begin{bmatrix} j \\ j-i, j-k, i+k-j \end{bmatrix},$$

$A(k, j, i, n) = \frac{(k+i-j)(2n+k+i-j+1)}{2},$
$B(k, j, i) \quad = \frac{(k+i-j)(k+i-j+1)}{2},$
$C(k, j, r, n) = (rk - j)n.$

Note that 3.1.1 agrees with the results in [1]. Also, the positivity of B and C in result 3.2 and 3.3 imply that $P_{k,j}^{(r)}$ are indeed polynomials.

4 Conjectures

After some experimentation with all of the binomial terms of the form $\prod_i \binom{n+d_i k}{b_i k + c_i}$, we found out that the only such terms that satisfy the condition (4) seem to be of the form $\binom{n}{k+c}\binom{n+k}{k+c}$ or $\binom{n+dk}{k}$ for any fixed integer c and d. We already saw that it was true for the former, but for the latter, it is only proved to be true for $d = 0, 1$, but we conjecture that it holds for all non-negative integers d as follows.

Conjecture 4.1. *For any integers d, $k \geq 0$ and $r \geq 1$, there exist integers $a_{d,k,j}^{(r)}$ such that*

$$\binom{n+dk}{k}^r = \sum_j a_{d,k,j}^{(r)} \binom{n+dj}{j} \quad \text{for all } n = 0, 1, 2, \ldots.$$

Moreover $a_{d,k,j}^{(r)}$ can be defined as following: $a_{d,k,k}^{(1)} = 1$, $a_{d,k,j}^{(1)} = 0$ for $j \neq k$ and

$$a_{d,k,j}^{(r+1)} = \sum_i S_d(k, j, i) a_{d,k,i}^{(r)},$$

where, $S_d(k, j, i)$ are integers, independent of r, for all d, k, j, i.

Conjecture 4.2. *For a fixed integer d, $S_d(k, j, i)$ defined above are holonomic but not of the first order, ie. no closed form solution, except $d = 0, 1$.*

5 Conclusions

We presented a motivated and streamlined new proof of the main result of [1], as well as two new, much deeper, identities, and made two conjectures. But the main interest of this paper is in illustrating a *methodology*, using computers (via *experimental mathematics*), to generate data, then formulate conjectures, and finally having the very same computer *rigorously* prove its own conjectures. We believe that this methodology has great potential almost everywhere in mathematics.

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