Chapter 13 Partial Derivatives

13.5 The Chain Rule

Recall chain rule from calculus I where $y = f(u)$ and $u = g(x)$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Now suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable function of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

<u>Example 1:</u> If $z = xy^2$, where $x = t^2$ and $y = \sqrt{t}$. Find $\dfrac{dz}{dt}$.

<u>Solutions:</u>
First way: Direct computation: $z = t^3$ then $\dfrac{dz}{dt} = 3t^2$.
Second way: Chain rule: $\dfrac{dz}{dt} = y^2(2t) + 2xy \cdot \dfrac{1}{2\sqrt{t}} = 2t^2 + t^2 = 3t^2$.

<u>Example 2:</u> If $z = xy + y^2$, where $x = \sin t$ and $y = e^{-t}$. Find $\dfrac{dz}{dt}$ when $t = 0$.

<u>More general case:</u> Let $z = f(x, y)$, $x = g(s, t)$, $y = h(s, t)$. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

<u>Example 3:</u> If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Application: Implicit Differentiation

Chain rule gives another way to solve the implicit differentiation. Let $F(x, y) = 0$. Then differentiate F both sides with respect to x using chain rule.

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{-F_x}{F_y}.$$

The following example can be done by applying this formula or by implicit differentiation with respect to x.

Example 4: Find $\dfrac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

Answer: $\dfrac{dy}{dx} = \dfrac{2y - x^2}{y^2 - 2x}$.

Similarly, for $z = f(x, y)$, we let $F(x, y, z) = 0$. Then

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

But $\dfrac{\partial x}{\partial x} = 1$ and $\dfrac{\partial y}{\partial x} = 0$. This equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

or

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}.$$

Example 5: Find $\dfrac{\partial z}{\partial x}$ if $yz = \ln(x + z)$.

13.6 Directional Derivatives and Gradients

Definition (Gradient Vector)**.** If f is a function of two variables, then the **gradient of** f is the vector function ∇f

$$\nabla f(x, y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle.$$

Example 1: Find the gradient vector of $f(x, y) = \sin x + e^{xy}$.

Definition (The directional derivative)**.** The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if the limit exists.

Then

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b = \nabla f(x,y) \cdot \mathbf{u}.$$

$D_{\mathbf{u}} f(x,y)$ is the rate of change of $f(x,y)$ in the direction $\mathbf{u}$. The partial derivatives in x and in y are $D_{\langle 1,0\rangle} f(x,y)$ and $D_{\langle 0,1\rangle} f(x,y)$.

Example 2: Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point $(x,y) = (2,-1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

Answer: $D_{\mathbf{u}} f(2,-1) = \dfrac{32}{\sqrt{29}}$.

Example 3: Find the directional derivative $D_{\mathbf{u}} f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ in the direction of angle $\theta = \dfrac{\pi}{6}$ to the x-axis. Also find $D_{\mathbf{u}} f(1,2)$.

Answer: $D_{\mathbf{u}} f(x,y) = \dfrac{1}{2}\left(3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y\right)$, $D_{\mathbf{u}} f(1,2) = \dfrac{13 - 3\sqrt{3}}{2}$.

Theorem 1. *The maximum value of $D_{\mathbf{u}} f(x,y)$ is $\|\nabla f(x,y)\|$.*

Proof. $D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u} = \|\nabla f(x,y)\| \|\mathbf{u}\| \cos\theta = \|\nabla f(x,y)\| \cos\theta$. $\square$

Example 4: Find the maximum value of $D_{\mathbf{u}} f(x,y)$ of the previous problem.