Chapter 13 Partial Derivatives

13.7 Tangent Planes

Given a surface S with equation $z = f(x, y)$. We find the **tangent plane** to the surface S at a given point P on the surface.

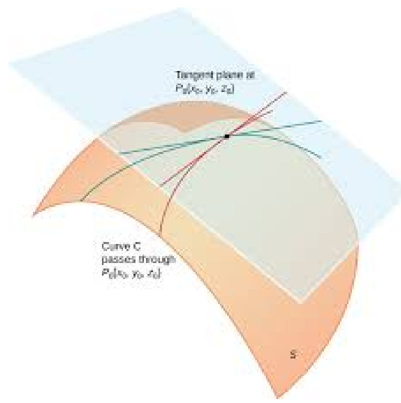


Figure 1: Tangent plane to the surface S at point P

To be precise, we consider any two curves C_1, C_2 on the surface S that pass through the point P. Let T_1 and T_2 be the tangent line of these curves at point P. The tangent plane is the plane that contains the lines T_1 and T_2.

Assume the tangent plane passing through the point $P(x_0, y_0, z_0)$ to be

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

where $\langle A, B, C \rangle$ is the normal vector $\mathbf{n}$ of this plane.

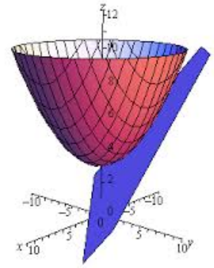
We then do implicit differentiation of this equation w.r.t. x and w.r.t. y to get

$$A = -Cf_x \qquad \text{and} \qquad B = -Cf_y.$$

Plug these back and simplify (assume $C \neq 0$) to get the formula of tangent plane:

$$z = z_0 + f_x(x - x_0) + f_y(y - y_0).$$

Example 1: Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.



<u>Answer:</u> $z = 4x + 2y - 3$.

Tangent plane of implicit function

Now given to us the surface $F(x, y, z) = c$. We want to find the tangent plane at the point $P(x_0, y_0, z_0)$.

Consider the curve C on this surface, $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. After differentiates on both sides of $F(x, y, z) = c$ with respect to t (chain rule), we have that

$$\bigtriangledown F \cdot \mathbf{r}'(t) = 0.$$

This means $\bigtriangledown F$ is perpendicular to any tangent vector at point P. Therefore $\bigtriangledown F$ is the normal vector of the tangent plane, i.e.

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

<u>Example 2:</u> Given the ellipsoid $x^2 + 4y^2 + z^2 = 18$.

a) Find the tangent plane to this ellipsoid at the point $(1, 2, 1)$.

b) Find the parametric equations of the normal line (line perpendicular to the plane) at this point.

c) Find the acute angle that the plane makes with the xy-plane.

<u>Answer:</u>

a) $2(x - 1) + 16(y - 2) + 2(z - 1) = 0$ or $x + 8y + z = 18$.

b) $x = t + 1, \quad y = 8t + 2, \quad z = t + 1 \to x - 1 = \dfrac{y - 2}{8} = z - 1$.

c) $\arccos(\dfrac{1}{\sqrt{66}})$.

Note: For the special case when the surface is written in the form $z = f(x, y)$. We can rewrite the relation as $0 = F(x, y, z) = z - f(x, y)$. We apply the previous formula to get

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

or

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

which agrees with the first formula of this section.

13.8 Maxima and Minima of Functions of Two Variables

The main application of differential calculus is to find max/min of a function. In this section, we investigate it for the case of function of two variables, $z = f(x, y)$. The first theorem is about the condition of local max/min.

Theorem 1. *If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exists there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.*
A point (a, b) is called a critical point.

Example 1: Find all the critical points of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Answer: $\dfrac{\partial f}{\partial x} = 2x - 2$ and $\dfrac{\partial f}{\partial y} = 2y - 6$. Then f has a local minima/maxima at $(x, y) = (1, 3)$.

Second Derivative Test

Similar to Calculus1, we need to find the way to check the critical point whether it gives local maxima, local minima or saddle point. We do this by applying the *second derivative test.*

Theorem 2 (Second Derivative Test)**.** *Suppose that the second partial derivatives of f is continuous on a disk with center (a, b), and suppose $f_x(a, b) = 0$ and $f_y(a, b) = 0$.*

Let $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$. *Then*

1. *If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.*

2. *If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.*

3. *If $D < 0$ then $f(a,b)$ is not a local minimum or minimum.*

We call the point (a,b) in case 3 a saddle point.

Example 2: Find the local maximum and minimum values and saddle points of

$$f(x,y) = x^4 + y^4 - 4xy + 1.$$

Answer: There are three critical points $(0,0), (1,1), (-1,-1)$.
$f(0,0)$ is a saddle point and $f(1,1), f(-1,-1)$ are local minimums.

Example 3: Find the shortest distance from the point $(1,0,-2)$ to the plane $x+2y+z=4$.

Answer: $\frac{5\sqrt{6}}{6}$. Note that this could be done by the formula from earlier.

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}} = \frac{|1-2-4|}{\sqrt{1+4+1}} = \frac{5}{\sqrt{6}} = \frac{5\sqrt{6}}{6}.$$

Absolute Maximum and Minimum Values

The absolute max/min are the points that can be found from set of all the critical points and points on the boundary.

Outline of the method:

1. Find the values of f at the critical points of f inside the region.
2. Find the extreme values of f on the boundary.
3. The largest value from step 1 and 2 is the absolute max. value.
   The smallest value from step 1 and 2 is the absolute min. value.

Example 4: Find the absolute maximum and minimum values of $f = 2x^2 + y^3$ where the region $D = \{(x,y)| \ x^2 + y^2 \leq 1\}$.

Answer: Step 1: Find the critical point inside the region $0 = f_x = 4x$ and $0 = f_y = 3y$. Therefore we have only one critical point at $(x,y) = (0,0)$. Step 2: The extreme value on the boundary $x^2 + y^2 = 1$. Plug this eq. back in f to get

$$f = 2 - 2y^2 + y^3$$

$0 = \dfrac{df}{dy} = -4y + 3y^2$. Then $y = 0$ or $y = 4/3$. Plug these values back to $x^2 + y^2 = 1$ to find x. For $y = 0$, we have $x = 1, -1$. For $y = 4/3$, there is no solution for x. Step 3: There are 3 candidates $(x, y) = (0, 0)$, $(1, 0)$ and $(-1, 0)$. The absolute maximum of f is 2 at $(x, y) = (1, 0)$ and $(-1, 0)$. The absolute minimum of f is 0 at $(x, y) = (0, 0)$.

Example 5: (Long!) Find the absolute maximum and minimum values of $f = 3 + xy - x - 2y$ on the closed triangular region with vertices $(1, 0), (5, 0)$, and $(1, 4)$.