Chapter11 Three-dimensional space, Vectors

11.1 Rectangular Coordinates in 3-Space; Spheres; Cylindrical Surfaces

Rectangular Coordinate System

Some demonstration on 1-space, 2-space and 3-space.
0-dimension: point and a point on the line
1-dimension: line on a plane
2-dimension: surface on a 3-dimensional space.

Three Dimensional Space

Midpoint between $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

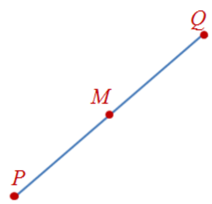

Figure 1: Midpoint between P and Q
(credit: http://proofsfromthebook.com/)

Distance from the origin to the point $P(x_0, y_0, z_0)$:

$$d = \sqrt{x_0^2 + y_0^2 + z_0^2}.$$

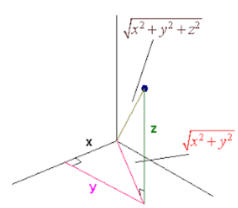

Figure 2: Distance from the origin to the point
(credit: quora.com)

Distance from P_0 to P_1 :

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}.$$

Example: Find the distance d between the points $(1, -2, 4)$ and $(-3, -6, -3)$.

Some Surfaces

1. Plane:
   $x = 0$ or $x = 3$ or $z = 5$,
   General form: $Ax + By + Cz = D$.

2. Cylindrical Surface:
   $x^2 = y + 3, \quad x^2 + z^2 = 5$.

3. Sphere:
   $(x - 1)^2 + (y + 2)^2 + (z + 6)^2 = 3^2$.
   General form: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = d^2$.

11.2 Vectors

Unlike the numbers, vectors have both length and direction (with no actual initial point). We use the notation $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as a basis for the vector in 3-dimensional space. $\mathbf{i}$ is a unit vector in the positive x-direction. $\mathbf{j}$ is a unit vector in the positive y-direction. $\mathbf{k}$ is a unit vector in the positive z-direction.
Some examples of vectors in 3-dimensions are $\mathbf{v} = 5\mathbf{i} + 3\mathbf{j}, \quad \mathbf{w} = -2\mathbf{j}, \quad \mathbf{0} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$.

Vector can be written as a linear combination of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ or in a bracket notation i.e.

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = \langle a, b, c \rangle.$$

Vector from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

Arithmetic operations on Vectors

If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, then

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle, \\ \mathbf{v} - \mathbf{w} &= \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle, \\ k\mathbf{v} &= \langle kv_1, kv_2, kv_3 \rangle. \end{aligned}$$

Rules of Vector Arithmetic

1. $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$,

2. $\mathbf{v}+(-\mathbf{v})=0$,

3. $\mathbf{w}=c\mathbf{v}, \quad c\neq 0 \iff \mathbf{w}//\mathbf{v}$.

Norm of a vector

$$\|\mathbf{v}\|=\sqrt{v_1^2+v_2^2+v_3^2}.$$

1. $\|c\mathbf{v}\|=c\|\mathbf{v}\|$.

2. Unit vector (vector of length one) of $\mathbf{v}$ is $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.

Examples:

1. Find the vector of length 2 that makes an angle of $\dfrac{\pi}{4}$ with the positive x-axis.

2. Find the angle that the vector $v=-\sqrt{3}\mathbf{i}+\mathbf{j}$ makes with the positive x-axis.

11.3 Dot product; Projections

Definition. Let $\mathbf{u}=\langle u_1,u_2\rangle$ and $\mathbf{v}=\langle v_1,v_2\rangle$ be vectors in 2d-space.
Then $\mathbf{u}\cdot\mathbf{v}$ is defined as

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2$$

Similarly if $\mathbf{u}=\langle u_1,u_2,u_3\rangle$ and $\mathbf{v}=\langle v_1,v_2,v_3\rangle$ then

$$\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2+u_3v_3.$$

Example: $\mathbf{u}=\langle 1,2,3\rangle, \mathbf{v}=\langle -3,-7,-1\rangle$ then $\mathbf{u}\cdot\mathbf{v}=-3-14-3=-20$.

Properties of Dot Product

a) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

b) $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$

c) $\mathbf{0}\cdot\mathbf{v}=\mathbf{0}$.

d) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$.

Angle between Vectors

Theorem 1. *Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ then*

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}.$$

Proof. Consider $\|\mathbf{v}-\mathbf{u}\|^2$, first by applying the law of cosines, $c^2 = a^2+b^2-2ab\cos\theta$. And second by applying property 4 of dot product. $\square$

Example: Find the angle between the vector, $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and

a) $\mathbf{v} = 3\mathbf{i} - \mathbf{k}$

b) $\mathbf{w} = -2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

Orthogonal Vector

The (non-zero) vector $\mathbf{u}$ and $\mathbf{v}$ are orthogonal $(\theta = \frac{\pi}{2}) \iff \mathbf{u}\cdot\mathbf{v} = 0$.