Chapter 11 Three-dimensional space, Vectors

11.3 Dot product; Projections

Definition. Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors in 2d-space.
Then $\mathbf{u} \cdot \mathbf{v}$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

Similarly if $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example: $\mathbf{u} = \langle 1, 2, 3 \rangle, \mathbf{v} = \langle -3, -7, -1 \rangle$ then $\mathbf{u} \cdot \mathbf{v} = -3 - 14 - 3 = -20$.

Properties of Dot Product

a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

c) $\mathbf{0} \cdot \mathbf{v} = \mathbf{0}$.

d) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

Angle between Vectors

Theorem 1. *Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ then*

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\| \|\boldsymbol{v}\|}.$$

Proof. Consider $\|\mathbf{v} - \mathbf{u}\|^2$, first by applying the law of cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$.
And second by applying property 4 of dot product. □

Example: Find the angle between the vector, $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and

a) $\mathbf{v} = 3\mathbf{i} - \mathbf{k}$

b) $\mathbf{w} = -2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

Orthogonal Vector

The (non-zero) vector $\mathbf{u}$ and $\mathbf{v}$ are orthogonal $(\theta = \dfrac{\pi}{2}) \iff \mathbf{u} \cdot \mathbf{v} = 0$.

11.4 Cross Product

Determinants

2-by-2 matrix: $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$

3-by-3 matrix: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$

Definition (Cross Product)**.**

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ then the cross product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example: Find $\mathbf{i} \times \mathbf{j}$.

Example: Let $\mathbf{u} = \langle 1, -2, 2 \rangle, \mathbf{v} = \langle 3, -1, 1 \rangle$ and $\mathbf{w} = \langle 1, 0, -2 \rangle$. Find

a) $\mathbf{u} \times \mathbf{w}$,

b) $\mathbf{w} \times \mathbf{u}$,

c) $\mathbf{v} \times \mathbf{v}$.

Properties of Cross Product

1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
2. $\mathbf{u} \times \mathbf{v} = 0 \iff \mathbf{u} // \mathbf{v}$.
3. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

   In symbol $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$ and $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$.
4. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin\theta =$ Area of parallelogram.

Remark: 1. and 2. can be shown via the property of determinant. 3. and 4. can be shown by direct calculation.

Summary

1. Dot product: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$,
   $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$.

2. Vectors are orthogonal (perpendicular) $\iff$ $\mathbf{u} \cdot \mathbf{v} = 0$.
   Vectors are parallel $\iff$ $\mathbf{u} = c\mathbf{v}$ for some $c \in \mathbb{R}$ or $\mathbf{u} \times \mathbf{v} = 0$.