Chapter 11 Three-dimensional space, Vectors

11.5 Parametric Equations of Lines

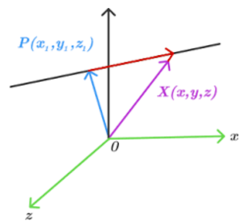

Figure 1: Line in 3-dimensional space
(credit: brilliant.org)

Line can be written as vector equation form or parametric equation form.

Vector Equation: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}, \quad \langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle.$

Parametric Equation: $L : x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$

Examples:

1. Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

2. Find a parametric equations for the line that passes through the point $(0, -2, 1)$ and $(1, 1, 1)$.

Parallel lines:
Given $\mathbf{r}_a = \mathbf{r}_0 + t\mathbf{v}_0$ and $\mathbf{r}_b = \mathbf{r}_1 + s\mathbf{v}_1$.
Then $\mathbf{r}_a // \mathbf{r}_b \iff \mathbf{v}_0 // \mathbf{v}_1 \iff \mathbf{v}_0 = c\mathbf{v}_1$ for some constant c.

Intersecting lines: Find the intersecting point between two lines:
$\mathbf{r}_1 = \langle 1, 1, 0\rangle + t\langle 1, -1, 2\rangle$ and $\mathbf{r}_2 = \langle 2, 0, 2\rangle + s\langle -1, 1, 0\rangle$.

Solution: set up three equations and solve for s and t.
$x : 1 + t = 2 - s, \quad y : 1 - t = s \quad z : 2t = 2.$

Skew lines: Skew lines are two lines that do not intersect and are not parallel.

Remarks

1. Another form for Lines with no extra variable t.
   Given $\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$.

<u>Symmetric Equation:</u> $\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$.

2. There is no unique way to write the straight line since variable t, $-\infty < t < \infty$ can be shifted by any constant.

11.6 Planes

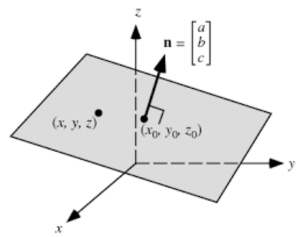

Figure 2: Plane in 3-dimensional space
(credit: mathworld.wolfram.com)

A vector $\mathbf{n}$ perpendicular(orthogonal) to a plane is called *normal vector*. Therefore any $\mathbf{n}$ perpendicular to any vector $\mathbf{r} - \mathbf{r}_0$ where $\mathbf{r}$ and $\mathbf{r}_0$ are any two points on the plane.

The equation of plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

which can also be written in the form

$$ax + by + cz = d, \quad a, b, c, d \in \mathbb{R},$$

given that $\mathbf{n} = \langle a, b, c\rangle$.

As long as the problem gives $\mathbf{n}$ and point $\mathbf{r}_0$, we can find the equation of the plane.

Example 1: Let $\mathbf{n} = \langle 2, -2, 1\rangle$ and $\mathbf{r}_0 = \langle -3, -4, 5\rangle$. Find the equation of the plane.

Solution: From $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$, we have
$\langle 2, -2, 1\rangle \cdot \langle x+3, y+4, z-5\rangle = 0$ which simplify to $2x - 2y + z = 7$.

Example 2: Find an equation of the plane that passes through the origin and parallel to the plane $2x - y + 3z = 1$.

Solution: Here $\mathbf{n} = \langle 2, -1, 3\rangle$ and $\mathbf{r}_0 = \langle 0, 0, 0\rangle$. Therefore, from $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$, the equation of the plane is $\langle 2, -1, 3\rangle \cdot \langle x-0, y-0, z-0\rangle = 0$ which simplify to $2x - y + 3z = 0$.

If the two lines on the plane are given, we can also find the equation of the plane.

Example 3:

a) Find the point which the given lines intersect:

$$\mathbf{r}_1 = \langle 1, 1, 0 \rangle + t\langle 1, -1, 2 \rangle$$

and

$$\mathbf{r}_2 = \langle 2, 0, 2 \rangle + s\langle -1, 1, 0 \rangle.$$

b) Find an equation of the plane that contains these lines.

Solution: a) Solve t and s from $1+t = 2-s, 1-t = -s$ and $2t = 2$. The solution is $t = 1, s = 0$. After plug t or s back, we know the intersection point is $\langle 2, 0, 2 \rangle$.

b) $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 1, -1, 2 \rangle \times \langle -1, 1, 0 \rangle = \langle -2, -2, 0 \rangle$. Any point on the plane for $\mathbf{r}_0$ would make the same equation. Here we pick $\mathbf{r}_0 = \langle 2, 0, 2 \rangle$. The equation of the plane is

$$0 = \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = \langle -2, -2, 0 \rangle \cdot \langle x-2, y-0, z-2 \rangle = -x + 2 - y.$$

It worths mentioning that given 3 points of the planes, we can also find the equation of the plane. We find the $\mathbf{v}_1$ and $\mathbf{v}_2$ from these points, and then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$. We could pick any point to be $\mathbf{r}_0$.

The following theorem gives the shortest distance from a point to the plane.

Theorem 1. *The distance D between a point (x_0, y_0, z_0) and the plane $ax + by + cz - d = 0$ is*

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

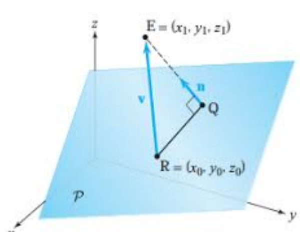

Figure 3: Shortest distance from a point to the plane
(credit: math.stackexchange.com)

Proof. Let $R = (x_0, y_0, z_0)$ be any point on the plane and $\mathbf{v} = \langle E - R \rangle$. Note that the normal vector $\mathbf{n}$ is $\langle a, b, c \rangle$ as usual. It follows that $|\mathbf{v} \cdot \mathbf{n}| = \|\mathbf{v}\|\|\mathbf{n}\|| \cos\theta| = \|\mathbf{n}\|D$.

Then

$$
\begin{aligned}
D = \frac{|\mathbf{v} \cdot \mathbf{n}|}{\|\mathbf{n}\|} &= \frac{|\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle \cdot \langle a, b, c \rangle|}{\|\langle a, b, c \rangle\|} \\
&= \frac{|ax_1 + by_1 + cz_1 - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.
\end{aligned}
$$

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