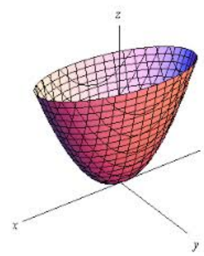
Chapter 11 Three-dimensional space, Vectors

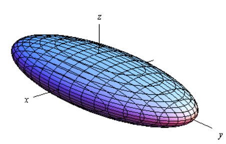
11.7 Quadric Surfaces

We show some typical surfaces $z = f(x, y)$ of polynomial total degree 2 in x and y. These surfaces will show up again in the integration section when we calculate the volume of the solids.

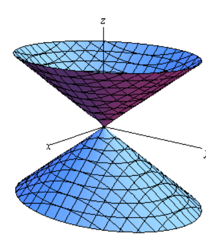
Elliptic Paraboloid: $z = x^2 + y^2$



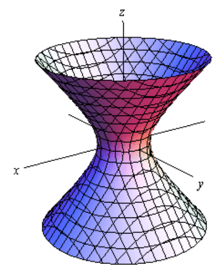
Ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



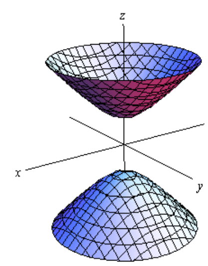
Cone: $z^2 = x^2 + y^2$



Hyperboloid of One Sheet: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$



Hyperboloid of Two Sheets: $\dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$



Credit for images in this section due to *http://tutorial.math.lamar.edu/*.

Chapter 12 Vector-Valued Functions

Section 12.1 Vector Functions and Space Curves

In this chapter, we talk about the vector valued functions in three dimensional space and the calculus that implement on it.

Curve in 3D: $\mathbf{r} = \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = \langle x(t), y(t), z(t)\rangle, \quad t \in \mathbb{R}.$

Example 1: $\mathbf{r}(t) = \langle -1, 2, 6\rangle + t\langle -5, 2, 1\rangle = (-1-5t)\mathbf{i} + (2+2t)\mathbf{j} + (6+t)\mathbf{k}$.

Example 2: $\mathbf{r}(t) = \langle t^3, \ln(5+t), \dfrac{\sin t}{t}\rangle$.

Most of the curves are very hard to draw or visualize. We will consider only the simple one.

Example 3: vector function in 2 dimension.
$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} = \langle \cos t, \sin t\rangle$.

Example 4: $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} = \langle \cos t, \sin t, t \rangle$.
This function is called "Helix".

Line Segment between two points: $\mathbf{r}(t) = (1-t)\mathbf{P} + t\mathbf{Q}, \quad 0 \leq t \leq 1.$

Example 5: Find a vector equation and parametric equations of the line segment that joins $P = (1, 3, -2)$ and $Q = (2, -1, 3)$.

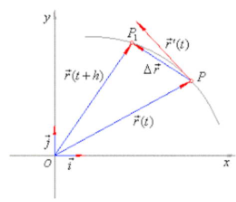
Section 12.2 Calculus of Vector Functions

Derivatives

Definition (Derivative of vector valued function $\mathbf{r}(t)$)**.**

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}.$$

Interpretation: $\mathbf{r}'(t)$ is a tangent vector of the curve.



Theorem 1. *Let* $\boldsymbol{r}(t) = \langle x(t), y(t), z(t) \rangle$*, where* $x(t), y(t), z(t)$ *are differentiable functions, then*

$$\boldsymbol{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

Example 1: Investigate the derivative of $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$.

In this case $\mathbf{r}(t)$ is always perpendicular to $\mathbf{r}'(t)$.
In general, if $\|\mathbf{r}(t)\| = c$ then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$. This is the property of sphere-tangent line.

The unit tangent line is defined as

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|},$$

where $\|\mathbf{r}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.

Example 2:

a) Find the derivative of $\mathbf{r}(t) = (1+t^3)\mathbf{i} + (te^{-t})\mathbf{j} + \sin 2t\mathbf{k}$.

b) Find the unit tangent vector at the point where $t = 0$.

Some properties of derivative of vector valued functions

1. $\dfrac{d}{dt}[\mathbf{u} \cdot \mathbf{v}] = \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'$
2. $\dfrac{d}{dt}[\mathbf{u} \times \mathbf{v}] = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$.

Integrals

Theorem 2. *Let $\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$, where $x(t), y(t), z(t)$ are integrable functions, then*

$$\int_a^b \boldsymbol{r}(t)\,dt = \langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt\rangle.$$

Example 3: Find the integral of $\mathbf{r}(t) = 2\cos t\mathbf{i} + \sin^2 t\mathbf{j} + 2t\mathbf{k}$.

Example 4: Evaluate the definite integral of $\int_0^{\frac{\pi}{2}} \langle \cos 2t, \sin 2t\rangle\, dt$.