Chapter 12 Vector-Valued Functions

12.3 Arc Length and Change of Parameter

Let L be the arc length of 2-dimensional or 3-dimensional space where t is from a to b.

2d-space: $\mathbf{r}(t) = \langle x(t), y(t) \rangle, \quad a \le t \le b,$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

3d-space: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \le t \le b,$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt.$$

Therefore $L = \int_a^b \|\mathbf{r}'(t)\| dt$.

Example 1: Find the length of the arc of the circular helix with vector equation

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

Answer: $2\sqrt{2}\pi$.

Example 2: Find the length of the arc of vector equation $\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from $(1, 0, 0)$ to $(1, 1, 1)$.

Answer: $\dfrac{1}{27}(13^{3/2} - 8)$.

We can also define arc length as a function which is called "arc length function".

$$s(t) = \int_a^t \|\mathbf{r}'(u)\| du.$$

The constant a is called the reference point. Equivalently $\dfrac{ds}{dt} = \|\mathbf{r}'(t)\|$.

Example 3: For circular helix, $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, we have $s(t) = \sqrt{2}t$.

Similarly, if we change a variable in the circular helix, $\mathbf{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$.

We note that $\left\|\frac{d\mathbf{r}}{ds}\right\| = 1$.