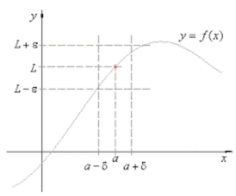
Chapter 13 Partial Derivatives

13.2 Limits and Continuity

Limits

We first consider the limit of a function of one variable,



The limit of function of one variable $f(x)$ is written as $\lim_{x\to a} f(x) = L$. We only consider the limit by approaching from the two directions, left or right.

Limit of functions of two variables is an analogous of those in one variable.

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Unlike one variable, the limit can be approach from infinitely many directions.

The goal is to show whether the limit exists or does not exist.

Example: $\displaystyle\lim_{(x,y)\to(1,2)} \sqrt{x^2+3+6y}$.
It is easy to see that the limit of this function exists.

Those one that are difficult to determine are the limits of the type $\dfrac{0}{0}$. We write up the solution differently depending whether the limit exists or not.

Case when the limit does not exist

Example 1: Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

In this example, if we take limit along the x-axis $\displaystyle\lim_{(x,0)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = 1$.

On the other hand if we take limit along y-axis $\displaystyle\lim_{(0,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = -1$.

Therefore the limit does not exist.

Example 2: Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Example 3: Show that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

In conclusion, if the limit does not exist, we have to find two directions in which the approaching values are different.

Case when the limit exists

Easy: Example: Find $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$.

Difficult: Example: Find $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$.

In the easy case, the function can be simplified and then be evaluated at the given point. The difficult case is, however, much more involved to show that the limit exists. We have to apply the $\epsilon - \delta$ argument. In this case, we will only do the easy case.

Continuity

Definition. A function f of two variables is called continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

To verify the continuity, we check 3 things:

1. f is defined at $(x, y) = (a, b)$.
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.
3. $f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$.

Example: Where is the function $g(x,y) = \arctan(\frac{y}{x})$ continuous?

Example: Where is the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ continuous?

Example: Is the function $f(x,y) = \begin{cases} \frac{x^4-y^4}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$ continuous at $(0,0)$?