Chapter 13 Partial Derivatives

13.3 Partial Derivatives

Definition (Partial derivatives with respect to x and y)**.**

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Notations: $f_x = f_x(x,y) = \dfrac{\partial f}{\partial x} = \dfrac{\partial f(x,y)}{\partial x} = \dfrac{\partial z}{\partial x}$.

Rule for finding partial derivatives of $z = f(x,y)$

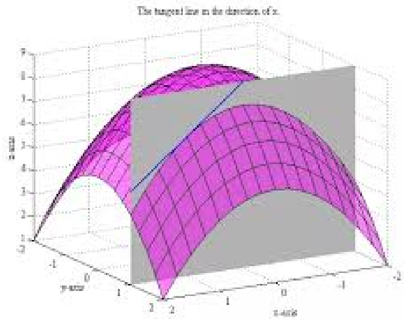
1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Example 1: If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

Example 2: If $f(x,y) = \sin\left(\dfrac{x}{1+y}\right)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

Partial Derivatives Viewed as Rates of Change and Slopes

$f_x(x_0,y_0)$ is the slope of the surface in the $x-$direction at (x_0,y_0).
$f_y(x_0,y_0)$ is the slope of the surface in the $y-$direction at (x_0,y_0).



$f_x(1,1)$ is the slop of tangent line in this picture.

<u>Example 3:</u> Let $f(x,y) = x^2y + 5y^3$.

a) Find the slope of the surface $z = f(x,y)$ in the x-direction at $(1,-2)$.

b) Find the slope of the surface $z = f(x,y)$ in the y-direction at $(1,-2)$.

Answers $a) - 4, \quad b) \, 61.$

Higher Derivatives

We can consider apply partial derivatives multiple number of times i.e. **the second partial derivatives** of f

$$(f_x)_x, \quad (f_x)_y, \quad (f_y)_x, \quad (f_y)_y.$$

<u>Example 4:</u> Find the second partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2.$$

Notations:

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f(x,y)}{\partial y \, \partial x} = \frac{\partial^2 z}{\partial y \, \partial x}.$$

Theorem 1 (Clairaut)**.** *If the functions f_{xy} and f_{yx} are both continuous on a disk D, then*

$$f_{xy}(a,b) = f_{yx}(a,b).$$

All of the functions we consider in this class are nice enough so that they always satisfy Caliraut's theorem.

Example 5: Laplace's equation is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Verify that $u(x, y) = e^x \sin(y)$ is a solution of Laplace's equation.

Implicit Differentiation

The implicit differentiation of functions of two variables can be done (similar to functions of one variable) through the partial derivative and chain rule.

Example 6: Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ where z is defined implicitly as a function of x and y by the equation:

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

13.4 Local Linearity and Differentials

We apply the idea of straight line approximations in both x and y directions to approximate the value of f.

Linearization of f at (a, b) is given by

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We then use $L(x, y)$ to approximate $f(x, y)$.

Example 1: Find linearization of $f(x, y) = xe^{xy}$ at $(1, 0)$. Then use it to approximate $f(1.1, -0.1)$.

Solution: $L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$.
Then $f(1.1, -0.1) \approx L(1.1, -0.1) = f(1, 0) + f_x(1, 0)(1.1 - 1) + f_y(1, 0)(-0.1 - 0) = 1$.
While $f(1.1, -0.1) = (1.1)e^{-0.11} \approx 0.98542$.

Differentials

Differential of function of one variable $dy \approx \Delta y = y_1 - y_2$. In Calculus I:

$$dy = f'(x)\, dx.$$

For function of two variables, $z = f(x, y)$, the differentials, dz, is

$$dz = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

The differentials here contains the same approximation idea as in the example above. dz is roughly $L(x, y) - f(a, b)$.

<u>Example 2:</u>

a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, Find dz.

<u>Answers:</u>
a) $dz = (2x + 3y)\, dx + (3x - 2y)\, dy$.
b) $dz = 0.65$. while the actual difference is $\Delta z = 0.6449$.

<u>Example 3:</u> If $z = 5x^2 + y^2$

a) Compute dz

b) Find the approximation if (x, y) changes from $(1, 2)$ to $(1.05, 2.1)$.

<u>Answers:</u>
a) $dz = 10x\, dx + 2y\, dy$
b) $dz = 10 \cdot 1(0.05) + 2 \cdot 2(0.1) = 0.9$.