Practice Problems 2 for Final Exam.

Mutivariable Calculus

1. a) Find the domain and range of $f(x, y) = x^2 e^{-\sqrt{y+1}}$.

   b) Compute the limit
   $$\lim_{(x,y)\to(0,0)} \frac{x^2 + 4y^2}{x^2 + 3y^2}.$$
   or prove that it does not exist.

   c) Compute the limit
   $$\lim_{(x,y)\to(0,0)} \frac{x^4 - 16y^4}{x^2 - 4y^2},$$
   or prove that it does not exist.

2. Let $f(x,y) = xe^{-y} + 5y$.

a) Compute f_x and f_y.

b) Find the slope of the surface $z = f(x,y)$ in the y-direction at the point $(2,5)$.

3. Let $L(x, y)$ denote the local linear approximation to

$$f(x, y) = x^2 + 2xy - 4x$$

at the point (1,2). Use $L(x, y)$ to approximate $f(1.01, 2.03)$.

4. a) Use the chain rule to find $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ if

$$f(x,y) = x^3 + y^3, \quad x = e^{u+v}, \quad y = 2u + 3v.$$

Express your answer in terms of u and v.

b) Given $z = f(x,y)$. Use implicit differentiation to find $\dfrac{\partial z}{\partial y}$ if

$$\ln(1+z) + xy^2 + 2z = 1.$$

5. a) Find the directional derivative of $f(x, y) = xe^y - ye^x$ at the point $P = (0, 0)$ in the direction $5\mathbf{i} - 2\mathbf{j}$

b) Find an equation for the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 25$$

at the point $P = (3, 0, 4)$.

6. Find all the local maximum and minimum values and saddle point(s) of the function $f(x, y) = 2xy - x^3 - y^2$.

Answers

1. a) Domain $= \{(x,y) \in \mathbb{R}^2 |\ y \geq -1\}$, Range $= \mathbb{R}^+ \cup \{0\}$.
   b) The limit does not exist. You could try $(x,y) \to (0,0)$ with $x = 0$ or try with $y = 0$. (something else would work too, like $x = y$, for example.)
   c) $\dfrac{x^4 - 16y^4}{x^2 - 4y^2} = x^2 + 4y^2$. So the limit is 0.

2. a) $f_x = e^{-y}$, $f_y = -xe^{-y} + 5$. b) Slope $= f_y(2,5) = -2e^{-5} + 5$.

3. $L(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 1 + 2(x-1) + 2(y-2)$.

   $f(1.01, 2.03) \approx L(1.01, 2.03) = 1.08$.

4. a) $\dfrac{\partial f}{\partial u} = 3e^{3(u+v)} + 6(2u+3v)^2$, $\dfrac{\partial f}{\partial v} = 3e^{3(u+v)} + 9(2u+3v)^2$.
   b) $\dfrac{\partial z}{\partial y} = \dfrac{-F_y}{F_z} = \dfrac{-2xy}{\frac{1}{1+z} + 2} = \dfrac{-2xy(1+z)}{3+2z}$.

5. a) $D_u(f) = \nabla f(0,0) \cdot \mathbf{u} = \langle 1, -1 \rangle \cdot \dfrac{\langle 5, -2 \rangle}{\sqrt{29}} = \dfrac{7}{\sqrt{29}}$.

   b) Equation of the tangent plane:

   $0 = F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 6(x-3) + 0(y-0) + 8(z-4) = 6x + 8z - 50$

   or $3x + 4z = 25$.

6. There are two critical points $(x,y) = (0,0)$ and $(x,y) = (2/3, 2/3)$.

   At $(0,0)$, $D = -4$ which implies $(0,0)$ is a saddle point.

   At $(2/3, 2/3)$, $D = 4$ and $f_{xx} = -4$ which implies $(2/3, 2/3)$ is a local maximum.