March 16, 2009

Lab 7: Gabriel's horn and Gabriel's wedding cake

TEAM MEMBERS

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INSTRUCTIONS: Work the following problems and write up your solutions neatly, clearly and carefully. All members of the team should understand and be able to explain the solutions.

Parts 1, 2 and 4 contain important information that you will need to investigate Gabriel's horn in part 3 and Gabriel's wedding cake in part 5.

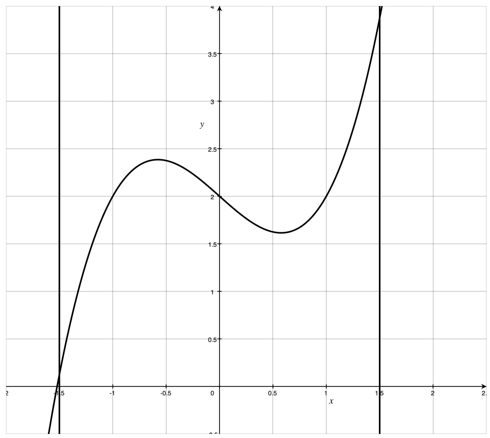
PART 1: Improper Integrals

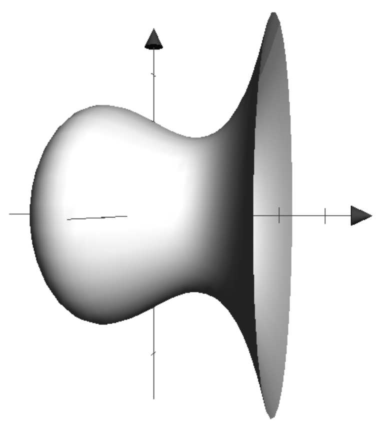
1. Compute $\int_1^\infty \frac{1}{x^2}\,dx$.

2. Compute $\int_1^\infty \frac{1}{x}\,dx$.

PART 2: Solids of Revolution

We create a solid of revolution as follows. Begin with a region in the plane. Below we see a region bounded by a curve $y = f(x)$ on the top, the x-axis on the bottom, and two x-values on the sides ($x = a$ and $x = b$). This region is then revolved about the x-axis to obtain a solid.





There is the formula that you will need later in this lab.

- The volume of the solid: $V = \int_a^b \pi f(x)^2 \, dx$

We will take a closer look at solids of revolution in sections 7.2 and 7.3.

PART 3: Gabriel's Horn

Suppose you own a company that manufactures *Gabriel's horns*. Gabriel's horn is the infinitely long brass instrument obtained revolving the curve $y = \dfrac{1}{x}$ $(x \geq 1)$ about the x-axis. Assume that the units on the x- and y-axes are feet. To build each horn you must create a plaster cast, then cover the cast with a thin layer of brass.[1]

1. Sketch Gabriel's horn.

2. Use the volume formula from part 2 to find an improper integral that gives the number of cubic feet of plaster needed to create the cast for Gabriel's horn.

[1]Gabriel's horn and its strange properties were discovered by Evangelista Torricelli, a student of Galileo's, in the seventeenth century.

3. Use your answer to the first question in part 1 to compute this integral. How much plaster is needed?

Note: Gabriel's horn is a beautiful example of the solid which has finite volume but infinite surface area. However the area of surface will be covered in Calculus 3.

PART 4: Infinite Series

Consider the famous infinite sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ (OK, not Brad Pitt famous, but some sums can be famous!) called the *harmonic series*. We would like to determine if this sum is finite or infinite.

1. Sketch the graph of $y = \dfrac{1}{x}$ for $x > 0$.

2. Estimate the area under this curve for $x \geq 1$ using Riemann sums. The rectangles should have width 1 and the heights should be determine using left-hand endpoints. Sketch the rectangles on the graph above. **Note:** there will be infinitely many rectangles! Write out the first five terms of this sum.

3. Is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ greater than or less than $\displaystyle\int_1^\infty \frac{1}{x}\, dx$?

4. What does this tell us about $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$?

Now we repeat the same process with the infinite sum $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots$

1. Sketch the graph of $y = \dfrac{1}{x^2}$ for $x > 0$.

2. Estimate the area under this curve for $x \geq 1$ using Riemann sums. The rectangles should have width 1 and the heights should be determine using **right-hand** endpoints. Sketch the rectangles on the graph above. Write out the first five terms of this sum.

3. What does this tell us about $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots$? In particular, is the sum finite or infinite?

We will investigate these and other infinite sums in great detail in Chapter 8.

PART 5: Gabriel's Wedding Cake

While not at work making exotic instruments, you enjoy baking. One of your favorite cakes to bake is called *Gabriel's wedding cake*. Gabriel's wedding cake is an elaborate circular layer cake. The first layer has radius 1' and height 1', the second layer has radius 1/2' and height 1', the third layer has radius 1/3' and height 1', and so on. (Yes, there are infinitely many layers!)

1. What is the volume of the first layer? The second layer? The third layer? The fourth layer?

2. Give a mathematical expression for the volume of the entire cake. Use the information from part 4 to show that the volume of the cake is finite.

3. Now it is time to frost the cake. You must frost the entire cake after it is assembled (but not the bottom of the cake). This includes the sides and the top. It turns out that it requires π square feet of frosting to coat the exposed parts of the tops of all layers. To see this, just look down on the cake from above—it is a circle of radius 1'.

   Now we must determine how much frosting it takes to coat the sides. How much is needed to coat the side of the first layer? The second layer? The third layer? The fourth layer?

4. Give a mathematical expression for the amount of frosting needed for the sides of all the layers. Then use the information from part 4 to show that this sum is infinite. Thus it takes an infinite amount of frosting to coat Gabriel's wedding cake.

Note: Gabriel's wedding cake is another beautiful example of the solid which has finite volume but infinite surface area.