How to beat Capablanca

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Abstract

We consider the chess problem of checkmating a king with a king and a rook on an $m \times n$ board at a specific starting position. We analyze the fastest way to checkmate.

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1 Introduction

The first thing Capablanca mentions in his book, *Chess Fundamentals*, is how to checkmate with rook, as in Figure 1.

Capablanca writes,

> In this position the power of the rook is demonstrated by the first move, Ra7, which immediately confines the black king to the last rank, ...

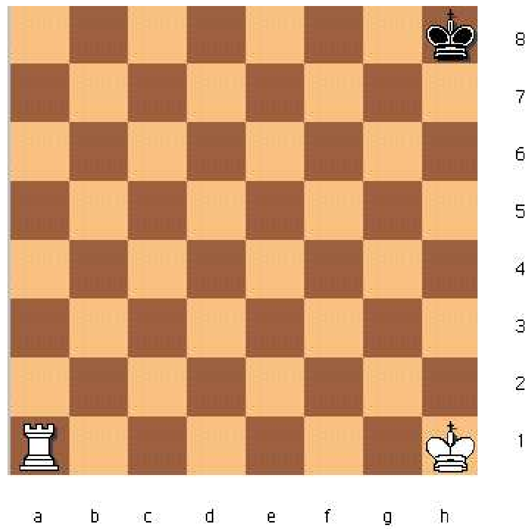

Figure 1: Starting position.

Capablanca did not give the fastest way to checkmate. With **1.Ra7**, the fastest way for White to checkmate Black king is 10 moves. While with **1.Rg1**, White can force a checkmate in 9 moves.

2 On an $m \times n$ board.

For the general $m \times n$ board with White king (WK) on $(m, 1)$, White rook (WR) on (1,1), and Black king (BK) on (m, n) where $m \geq 4$ and $n \geq 5$, we define $FM(m, n)$ to be the smallest number of moves for white to checkmate Black king.

Theorem. $FM(m,n) = \begin{cases} n & \textit{if } n \textit{ is odd;} \\ n+1 & \textit{if } n \textit{ is even.} \end{cases}$

Let's call the right side $G(m, n)$. We will prove this by
1) Showing a sequence of White moves that force the checkmate in $G(m, n)$ moves.
2) Showing that Black has a strategy to survive up to $G(m, n) - 1$ moves.

Lemma 1. *In the given postion on an $m \times n$ board, White can give a checkmate in n moves if n is odd.*

I will give the sequence of white moves so that for all the choices of Black moves, white can give the checkmate in n moves.

First move: 1.WR(m-1,1) BK(m,n-1) (only move)

Then White can make a sequence of White king moves up one square no matter what Black responses are. Black's only responses are moving the king up and down along the m^{th}column. But this can not interrupt White king because of the parity. It will take $n-4$ moves for white king to move up to the square $(m, n-3)$. We have Figure 2 below with White to move.

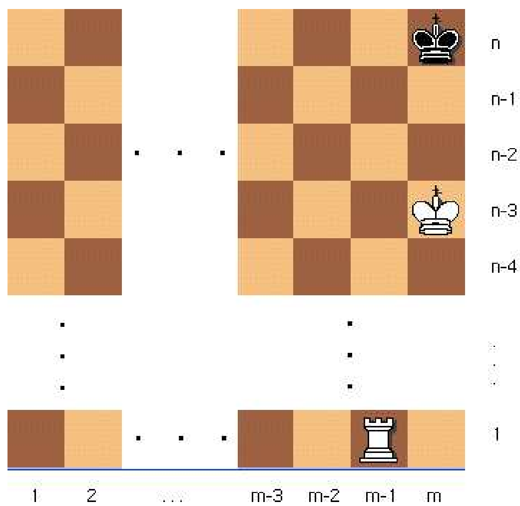

Figure 2: Position before checkmate.

White can finish this off by playing the forcing move

1. WK(m-1,n-2) BK(m-1,n) (only move)
2. WR(m-2,1) BK(m,n) (only move)
3. WR(m-2,n) Checkmate

The total number of moves is $1 + (n-4) + 3 = n$.

Lemma 2. *White can force a checkmate in $n+1$ moves, if n is even.*

The strategy is almost the same as in Lemma 1. However, because of the parity we have to make one waiting move, i.e.,WR(m-1,2) while White king tries to move upward. Since $n \geq 5$, we have enough room for this plan.

Note: The first move WR(m-1,1) with the strategy above gives White the fastest way to mate. As we can see it requires at least $n-3$ (resp. $n-2$ moves) for n odd (resp. even) for White King to move up the board and at least 2 more Rook moves before White can force a checkmate.

Lemma 3. *Black has a strategy to survive up to $n-1$ moves, if n is odd.*

The general plan for Black to survive is to move to the middle of the board as much as possible. At some point White has to move his Rook to restrict the possible moves of Black king. Then he must use White king to push Black king to the edge of the board. In addition, the sooner the Rook moves, the better. Therefore we will assume the first move is a Rook move. Note that the only mating positions are when Black king is at the edge of the board. Now we consider 2 cases of the first Rook move:

Case 1 The first rook move is vertical (moving along a column).

In this case, Black tries to move to the middle of the board as much as possible. Once Black king gets there, he will try to stay there as long as possible before he is forced to the corner. Below is an example where the size of the board is 9×9.

1. WR(1,8)	BK(8,9)
2. WK(8,2)	BK(7,9)
3. WK(7,3)	BK(6,9)
4. WK(6,4)	BK(5,9)
5. WK(5,5)	BK(6,9)
6. WK(5,6)	BK(5,9)
7. WK(4,7)	BK(6,9)

8. WK(5,7) BK(7,9)
9. WK(6,7) BK(8,9)
10.WK(7,7) BK(9,9)
11.WK(8,7) BK(8,9)
12.WR(1,9) Checkmate

On $m \times n$ board, we can see that White has to move his Rook twice, and White king moves up the board in $n-3$ moves and chases Black king back to the corner with at least another $\lceil (\frac{m}{2}) \rceil - 1$ moves. For example on a 9×9 board, it takes $2 + (9-3) + (5-1) = 12$ moves. This is the best that White can do. Therefore $F(m,n) = 2 + (n-3) + (\lceil (\frac{m}{2}) \rceil - 1) \geq n$ moves (since $m \geq 4$).

Case 2 The first Rook move is horizontal (moving along a row).
(We exclude the move $R(m-1,1)$ since we already know that in that case the fastest number of moves to mate is n.)

Black king will try to move down and toward the middle of the board as much as possible. Once he gets blocked by the rook and the White king, Black king moves along the row (otherwise Black King will just try to stay in the middle of the board). He could move along the row since the Rook is not at column $m-1$. Figure 3 illustrates the situation:

1. ... BK(9,6) Now white has two choices: 2.WK(7,5) or 2.WR(7,6). For 2.WK(7,5), Black's response will be BK(8,7) and the best white response is 3.WK(8,5). In this case, White King needs one extra move to get to the row $n-3$.

Overall, White has to move the rook twice (the first rook move and the checkmate move); the king takes $n-3$ moves to go up the board and at least an extra move from king or rook of the two choices above. This makes the least possible number of moves to checkmate at least $\geq 2+(n-3)+1 = n$.

Lemma 4. *Black has a strategy to survive up to n moves, when n is even.*

The general plan in this case is similar. The strategy in Case 1 also works here. In Case 2, Black now has an advantage of parity. This will give him an extra move to survive when he faces White king in the middle of the board.

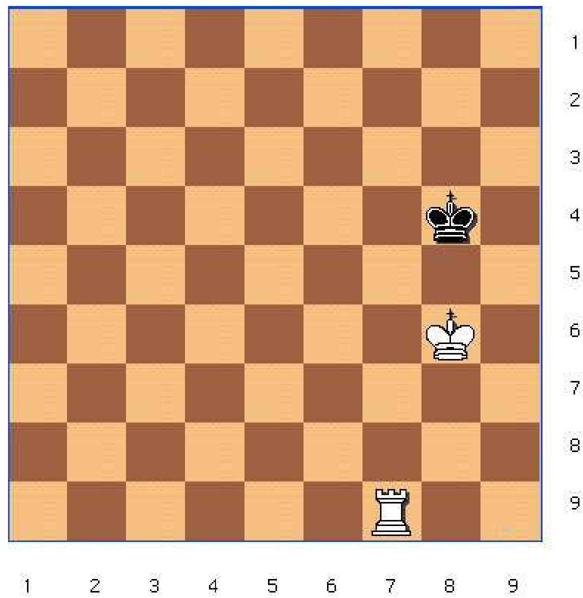

Figure 3: Intermediate position.

The details of the proof are left as an exercise to the reader.

By Lemmas 1, 2, 3 and 4, we conclude the theorem.
The induction method using to solve the above position could be later on generalized to more complicated starting postion with different pieces on the board.

3 About José Raul Capablanca

Jose Raul Capablanca was born in Havana, Cuba on November 19, 1888. He is regarded as one of the most gifted chess players of all time. According to Capablanca, at the age of four, he learned the rules of chess by watching his father play chess with a friend. In 1921, he won the world champion title from mathematician and chess player Emanuel Lasker, who held the title for 27 years. Capablanca was the third official world chess champion. He lost his title to Alexander Alekhine in 1927 and died in 1942.

The move that Capablanca suggested is Case 1 of Lemma 4. Black could survive at least $2 + (n-3) + (\lceil(\frac{m}{2})\rceil - 1) = 2 + 5 + 3 = 1$.

4 Acknowledgements

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References

[1] J.R. Capablanca, *Chess Fundamentals*, Everyman Chess, 1999. First published in 1921.