NO FEEDBACK? NO WORRIES! The art of guessing the right card

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Abstract

In 1998, Ciucu published "No-feedback card guessing for dovetail shuffles", an article which gives the optimal guessing strategy for n cards (n even) after k riffle shuffles whenever $k > 2\log_2(n)$. We discuss in this article the optimal guessing strategy and the asymptotic (in n) expected number of correct guesses for any fixed $k \ge 1$. This complements the work achieved two decades ago by Ciucu.

Keywords: card guessing; no feedback; optimal strategy; discrete probability; generating function.

The art^{math} of guessing it right

Over the past several decades, there have been studies and established theories attempting to tackle many aspects of the CARD GUESSING GAME, e.g. finding the optimal or worst case strategies, identifying the distribution and the expected number of correct guesses, etc. You might have seen a dealer performing "riffle shuffles", perhaps at the Vegas casinos or at the basement poker nights. Before play can begin, the cards have to be shuffled multiple times to mix the cards randomly. [Bayer and Diaconis (1992)] proved the famous result that after giving the deck seven riffle shuffles, all the 52 cards will be equally likely to be in one position as in another.



Card Guessing Game!

START: To start, you will need:

- A dealer and a player
- 1 deck of mn cards ordered 1, ..., 1, ..., n, ..., n

m copies m copies

How to Play?

- The dealer shuffles the deck according to some Shuffling Rule $\mathcal{R}.$
- Cards are then dealt from the top, and the player guesses each card one at a time.
- Play continues until the entire deck has been gone through.

SCORING: The player earns 1 point for each correct guess.

FEEDBACK: The player is given FEEDBACK $\mathcal{F} \in \{\text{COMPLETE, PARTIAL, No}\}$ after each guess.

OBJECT OF THE GAME: The object of the game is to earn as many points as possible.

Let us take a look at a quick guide of how the game is played. As shown above, the player may or may not receive feedback from the dealer depending on the rule. The complete feedback game is similar to blackjack for which the dealer reveals the card after each play (guess). The player thus can adjust their strategy of play, to improve his odds of guessing the next cards correctly, according to the cards they have already seen. For the no-feedback game, however, as no feedback is given, the player has to guess all the cards beforehand. To increase the probability of guessing the correct number, it is possible to guess the same card number for cards in different positions. Summary of terminologies and definitions related to the card guessing games are now given.

Terminologies

- LEVEL OF FEEDBACK: The player in the NO FEEDBACK variant is informed nothing about the correctness of his or her guess after each guess. In the COMPLETE FEEDBACK variant, on the other hand, the value of the card will be revealed to the player after each guess. Finally, in the PARTIAL FEEDBACK variant (also known as YES/NO FEEDBACK), the player is informed whether his or her guess was correct after each guess, but the value of the card will not be revealed in case of incorrect guesses.
- CARD SHUFFLING: If UNIFORM SHUFFLE is applied, the shuffled deck starts in a randomly shuffled state. Applying TOP TO RANDOM SHUFFLE means the dealer takes the top card and places it back into the deck at position i with probability p_i . For RIFFLE SHUFFLE (also known as GSR SHUFFLE or DOVETAIL SHUFFLE), the dealer cuts the deck into two piles before interleaving the two back into one.

Beginning with an ordered set (1 increasing subsequence), a single riffle shuffle will lead to a permutation with at most 2 increasing subsequences. The parameter k for the riffle and top to random shuffles represents the number of times the deck has been shuffled. In particular, k = 1 means a single shuffle.

• OPTIMAL (BEST) VERSUS OPTIMAL MISÈRE STRATEGY: A given strategy is said to be OPTIMAL (OR BEST) STRATEGY if the strategy achieves the maximum expected number of correct guesses, where the maximum is taken over all strategies. OPTIMAL MISÈRE (OR WORST-CASE) STRATEGY, which achieves the minimum expected number, is defined analogously [Diaconis et al. (2020)].

We summarize the game variants in the literature, characterized by the shuffling rules, the level of feedback, and the number of copies of each card type in Table 1.

| Authors | Shuffling \mathcal{R} | #Copies | Feedback \mathcal{F} | Contribution |
|------------------------------|-------------------------|---------|------------------------|---|
| [Diaconis and Graham (1981)] | uniform | 1, m | complete $(m > 1)$, | distribution of the number |
| | | | partial $(m = 1)$ | of correct guesses, |
| | | | | optimal and misère strategies |
| [Diaconis et al. (2020)] | uniform | m | partial | expected number of |
| and [Diaconis et al. (2022)] | | | | correct guesses |
| [Bayer and Diaconis (1992)] | k- riffle | 1 | complete | a sharp bound on the number |
| | | | | of shuffles to random deck |
| [Ciucu (1998)] | k- riffle | 1 | no | best strategy for $k = 1$ |
| | | | | and $k > 2\log_2(n)$, n even, |
| | | | | expected number of correct guesses |
| ► Present work | k- riffle | 1 | no | best strategy for any fixed $k \ge 1$, |
| | | | | expected number of correct guesses |
| [Pehlivan (2010)] | k- top to | 1 | no | best strategy for $k > 4n \log(n) + cn$, |
| | random | | | n even |
| [Liu (2021)] | 1- riffle | 1 | complete | expected number of |
| | | | | correct guesses |
| [Krityakierne et al. (2021)] | 1- riffle | 1 | complete | (refined) expected number |
| | | | | of correct guesses, higher moments |

Table 1: Variants of the CARD GUESSING GAME

Our contribution

As specified in the table, our contribution in this paper builds on and fills gaps of the no-feedback card guessing game with riffle shuffles in the literature. Specifically, [Ciucu (1998)] proved that, given a deck of n cards (n even), the best no-feedback guessing strategy after k riffle shuffles when $k > 2 \log_2(n)$ is to guess card number 1 for the first half and card number n for the second half of the deck. The expected number of correct guesses in this case is therefore at most 2. By using this strategy, Ciucu showed that after shuffling the cards the number of order $\log_2(n)$ times, the deck is well mixed, i.e. close to randomly distributed deck. In addition, Ciucu also gave the strategy for the other extreme case, k = 1. In this work, we focus exclusively on the analysis of games for a general but finite value of $k \ge 1$, which complements work by [Ciucu (1998)] in the no-feedback card guessing game with riffle shuffle variant.

Time to play

Let us consider now a strategy of guessing the cards after the deck is riffle-shuffled k-times, for $k \ge 1$, following these rules:

- At the beginning, a deck of n cards is ordered $1, 2, 3, \ldots, n$.
- The dealer riffle shuffles the deck k times.
- No feedback is given after each guess.

Let's start with k = 1, a single riffle shuffle

Take out a deck of n cards, cut it into two piles (possible to have 0 cards in one of the piles), and interleave the cards from two piles in any possible way. What are all possible permutations you can generate?



Figure 1: Examples of all possible permutations after 1-time riffle shuffling for n = 1, 2, 3, 4. For each card position, the color indicates the most likely number.

For instance, when n = 1, $\{1\}$ is the only possible permutation, but with multiplicity 2, generated from the piles ($\{1\}, \{\}$) or ($\{\}, \{1\}$). In general, there are 2^n possibilities, among which n + 1 are the identity permutation (having one increasing subsequence), and the

others are of two increasing subsequences (with multiplicity 1). We give a few examples in Figure 1. For each card position, the color indicates the most likely number to show up in that position.

What number is most likely to show up in position *i*?

We record the frequency of a card in position *i* being number *j* in the matrix $M = (m_{i,j})$. Then, the row *i* of the matrix corresponds to position *i* of the card. The column *j* of the matrix corresponds to card number *j*. For example $m_{1,3}$ gives the number of times the top card is 3.

When n = 4, the matrix M is given by

$$M = \begin{bmatrix} 9 & 3 & 3 & 1 \\ 4 & 6 & 4 & 2 \\ 2 & 4 & 6 & 4 \\ 1 & 3 & 3 & 9 \end{bmatrix}.$$

Probability matrix for k = 1 shuffle, $P^{(1)}$

For a fixed n, we define a probability matrix for k = 1 shuffle, $P^{(1)} := M/2^n$. Let $a(i, j, n) := \frac{m_{i,j}}{2^n}$ be the (i, j) element of $P^{(1)}$. Then, a(i, j, n) represents the probability of a card in position i being number j. It turns out that a(i, j, n) has an exact formula:

$$a(i,j,n) = \frac{1}{2^{i}} \binom{i-1}{j-1} + \frac{1}{2^{n-i+1}} \binom{n-i}{j-i}.$$
(1)

This formula was mentioned and proved earlier in [Ciucu (1998)]. We notice that the first term is non-zero when $i \ge j$ and the second term is non-zero when $j \ge i$. The row sum and column sum are both equal to 1, i.e. $P^{(1)}$ is a *doubly stochastic matrix*. This is easy to show in one direction, but not so easy to show in the other direction:

$$\sum_{j=1}^{n} a(i,j,n) = \frac{1}{2^{i}} \sum_{j=1}^{i} \binom{i-1}{j-1} + \frac{1}{2^{n-i+1}} \sum_{j=i}^{n} \binom{n-i}{j-i} = \frac{1}{2} + \frac{1}{2} = 1,$$

and

$$\sum_{i=1}^{n} a(i,j,n) = \sum_{i=j}^{n} \frac{1}{2^{i}} \binom{i-1}{j-1} + \sum_{i=1}^{j} \frac{1}{2^{n-i+1}} \binom{n-i}{j-i} = 1.$$

Also, the symmetric property is easily observed through the matrix $P^{(1)}$:

$$a(i, j, n) = a(n + 1 - i, n + 1 - j, n).$$
(2)

Guessing strategy exists. Do not blind guessing!

As we now have a formula for the probability that each number j shows up in the position i stored in the matrix $P^{(1)}$, it is possible to find the *optimal guessing strategy* that gives the maximum expected number of correct guesses. Since the first row of $P^{(1)}$ corresponds to the top card, the second row the second card, and so on, it is easily shown that one should pick the index of the largest value from each row i as a best guess for the card in the position i, for $i = 1, \ldots, n$.

The expected number of correct guesses

Let $X^{\mathcal{G}}$ be a random variable representing the number of correct guesses of an *n*-card deck if the player uses strategy \mathcal{G} . To find the expectation $E[X^{\mathcal{G}}]$, we consider

$$X_i^{\mathcal{G}} = \begin{cases} 1 & \text{if the player guesses the } i\text{th position correctly} \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$E\left[X^{\mathcal{G}}\right] = \sum_{i=1}^{n} E\left[X_i^{\mathcal{G}}\right] = \sum_{i=1}^{n} P\left(X_i^{\mathcal{G}} = 1\right) \le \sum_{i=1}^{n} \max_{1 \le j_i \le n} a(i, j_i, n),$$

the optimal strategy can be defined as follows.

Optimal guessing strategy \mathcal{G}^*

i, and in this case,

For a fixed row i $(1 \le i \le n)$ of the matrix $P^{(1)}$, let $j_i^* := \arg \max_j a(i, j, n)$. Under the optimal guessing strategy \mathcal{G}^* , the player guesses number j_i^* for the card position

$$E\left[X^{\mathcal{G}^*}\right] = \sum_{i=1}^n a\left(i, j_i^*, n\right).$$

For example, when n = 4, the largest number from each row *i* of the matrix *M* (those numbers having a small square around) tells us that $j_1^* = 1$, $j_2^* = 2$, $j_3^* = 3$, and $j_4^* = 4$. Thus, the optimal strategy is to guess number 1, 2, 3, 4, respectively.

In general, as we shall see later in our main theorem (Theorem 9, in its simplest case) that for a 1-time shuffled deck, the optimal strategy is to guess the top half of the deck with sequence

$$1, 2, 2, 3, 3, 4, 4, \ldots$$

and guess the bottom half in the reverse manner, i.e.

$$\dots, n-3, n-3, n-2, n-2, n-1, n-1, n.$$

To end the section, we restate a result from [Ciucu (1998)] regarding the value of $E[X^{\mathcal{G}^*}]$ for k = 1. The main result of our paper is to obtain similar results, but for any $k \ge 1$ times riffle shuffles.

Theorem 1 (Ciucu, 1998). For a deck of n cards, given that the no-feedback guessing of the whole deck after 1-time shuffle goes according to the best strategy, the average number of correct guesses is $\frac{2\sqrt{n}}{\sqrt{\pi}} + \mathcal{O}(1)$.

1 Preview: the *k*-time riffle shuffles

We now turn to a general case in which a deck of n cards is given k > 1 times riffle shuffles. Our goal is to obtain a similar statement to Theorem 1 for asymptotic n.

Probability matrix for general k shuffles, $P^{(k)}$

Analogous to a(i, j, n) for the case k = 1, we let $a_{i,j}^{(k)}$ denote the probability that the card number j will show up at position i after giving the deck k riffle shuffles (where we now make i, j subscripts, and suppress n from $a_{i,j}^{(k)}$ for simplicity). The probability matrix $P^{(k)}$ for k shuffles is subsequently defined by $(P^{(k)})_{i,j} = a_{i,j}^{(k)}$.

Since the transpose matrix $(P^{(1)})^T$ is a probability transition matrix, it follows that $P^{(k)} = (P^{(1)})^k$, the k-th power of the matrix, and so

$$a_{i,j}^{(k)} = \sum_{s_1=1}^n \cdots \sum_{s_{k-1}=1}^n a_{i,s_1}^{(1)} a_{s_1,s_2}^{(1)} \dots a_{s_{k-1},j}^{(1)}.$$
 (3)

Similar to the case k = 1, the expected number of correct guesses under the optimal strategy \mathcal{G}^* is

$$E\left[X^{\mathcal{G}^*}\right] = \sum_{i=1}^n a_{i,j_i^*}^{(k)}.$$
 (4)

Thus, finding the optimal guessing strategy boils down to determining the optimal strategy j_i^* which maximizes $a_{i,j}^{(k)}$ for each i = 1, ..., n.

Caution! Even though we have a closed-form expression for $a_{i,j}^{(1)}$ as given in Equation 1, simply plugging the closed-form formula in Equation 3 in attempt to solving for the maximizer of $a_{i,j}^{(k)}$ directly is unfortunately not tractable.



Before we dive into the details of the proof, we give an overview of the proof procedure, together with the accompanying Figure 2 depicting the p.m.f. $a_{60,j}^{(3)}$ (i.e. i = 60, k = 3) for n = 200 cards. Here, we want to find the maximizer index j_{60}^{*} of the p.m.f. In practice, the expression of the p.m.f. (the red line) is not tractable. We show later that the p.m.f. is indeed a mixture model with 2^k components, whose expressions are tractable via the generating functions.

Proof procedure

SETUP: We fix row *i* of matrix $P^{(k)}$, and let J_i be a random variable whose probability mass function (p.m.f.) is given by

$$P(J_i = j) = a_{i,j}^{(k)}, \text{ for } j = 1, \dots, n.$$

GOAL: Find the index j_i^* that maximizes $P(J_i = j)$.

PROCEDURE: The main steps in the proof proceed as follows:

1. Decompose the p.m.f. of J_i into a mixture model with 2^k components:

$$P(J_i = j) = \frac{1}{2^k} \sum_{t=1}^{2^k} p_{L_t}(j).$$

2. Define, for each p_{L_t} , the probability generating function (p.g.f.):

$$F_{L_t}(x) = \sum_{j=1}^n p_{L_t}(j) \cdot x^j.$$

- 3. Recover the distribution p_{L_t} through the p.g.f. $F_{L_t}(x)$.
- 4. Find, for n large, the location and the height of the peak of p_{L_t} .
- 5. Show, for n large, that the components $p_{L_1}, \ldots, p_{L_{2^k}}$ of the mixture model are (almost) non-overlapping.

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MAXIMIZER: The location of the highest peak among the 2^k components is the solution j_i^* .

2 Game start: the generating function of p_L

As outlined in the proof procedure, we let i = 1, ..., n be fixed, and focus exclusively on the *i*th row of the matrix $P^{(k)}$ whose p.m.f. is given by

$$P(J_i = j) = a_{i,j}^{(k)}, \ j = 1, \dots, n$$



Figure 2: Example (k = 3, n = 200, i = 60): (a) the p.m.f. of the mixture model J_{60} ; (b) the decomposed components of the mixture model, $P(J_{60} = j) = \frac{1}{8} \sum_{t=1}^{8} p_{L_t}(j)$

We start by decomposing $a_{i,j}$ (k = 1):

$$a_{i,j} = \frac{1}{2} [b_{i,j}^{(1)} + b_{i,j}^{(2)}], \tag{5}$$

where

$$b_{i,j}^{(1)} := \frac{1}{2^{i-1}} \binom{i-1}{j-1}, \qquad b_{i,j}^{(2)} := \frac{1}{2^{n-i}} \binom{n-i}{j-i}.$$

In general, for $k \ge 1$, we decompose $a_{i,j}^{(k)}$ into the sum of 2^k possible products of $b_{i,j}^{(l)}$. That is, we write

$$a_{i,j}^{(k)} = \frac{1}{2^k} \sum_{L} p_L(j), \tag{6}$$

where L is the list of length k whose entries are either 1 or 2,

$$p_L(j) := \sum_{s_1=1}^n \cdots \sum_{s_{k-1}=1}^n b_{i,s_1}^{(L[1])} b_{s_1,s_2}^{(L[2])} \cdots b_{s_{k-1},j}^{(L[k])},$$
(7)

and L[l] refers to the *l*th element of L.

For example, for k = 2,

$$a_{i,j}^{(2)} = \frac{1}{4} \left[p_{[1,1]}(j) + p_{[2,1]}(j) + p_{[1,2]}(j) + p_{[2,2]}(j) \right],$$

where

$$p_{[1,1]}(j) = \sum_{s=1}^{n} b_{i,s}^{(1)} b_{s,j}^{(1)} = \sum_{s=1}^{n} \frac{1}{2^{i-1}} \binom{i-1}{s-1} \frac{1}{2^{s-1}} \binom{s-1}{j-1},$$

$$p_{[2,1]}(j) = \sum_{s=1}^{n} b_{i,s}^{(2)} b_{s,j}^{(1)} = \sum_{s=1}^{n} \frac{1}{2^{n-i}} \binom{n-i}{s-i} \frac{1}{2^{s-1}} \binom{s-1}{j-1},$$

$$p_{[1,2]}(j) = \sum_{s=1}^{n} b_{i,s}^{(1)} b_{s,j}^{(2)} = \sum_{s=1}^{n} \frac{1}{2^{i-1}} \binom{i-1}{s-1} \frac{1}{2^{n-s}} \binom{n-s}{j-s},$$

$$p_{[2,2]}(j) = \sum_{s=1}^{n} b_{i,s}^{(2)} b_{s,j}^{(2)} = \sum_{s=1}^{n} \frac{1}{2^{n-i}} \binom{n-i}{s-i} \frac{1}{2^{n-s}} \binom{n-s}{j-s}.$$

Another example for k = 3, L = [1, 1, 2] leads to

$$p_L(j) = \sum_{s_1=1}^n \sum_{s_2=1}^n b_{i,s_1}^{(1)} b_{s_1,s_2}^{(1)} b_{s_2,j}^{(2)} = \sum_{s_1=1}^n \sum_{s_2=1}^n \frac{1}{2^{i-1}} \binom{i-1}{s_1-1} \frac{1}{2^{s_1-1}} \binom{s_1-1}{s_2-1} \frac{1}{2^{n-s_2}} \binom{n-s_2}{j-s_2}.$$

Generating function of p_L

The expression of p_L in Equation 7 initially involves (k-1)-nested binomial sum, for which we have no means of simplifying. The crux of our proposed approach is to find the generating function of p_L , giving rise to k-nested sum. After reordering the summation by moving the outermost sum (the summation over index j) to innermost, and evaluating the summations from the inside (that of the index j) to the outside (those of s_{k-1}, \ldots, s_1), fortunately, we obtain a very simple generating function for all $k \geq 1$.

We define the generating function by

$$F_L(x) = \sum_{j=1}^{n} p_L(j) \cdot x^j.$$
 (8)

Remark 1. $F_L(x)$ is a valid probability generating function as $F_L(1) = 1$. This also ensures that each p_L in Equation 6 is a valid probability mass function.

Let us look at some examples of $F_L(x)$:

• for k = 0,

$$F_{[\]}(x) = x^i.$$

This is a trivial case. When cards are not shuffled, the probability that the card in position i is i is 1.

• for k = 1,

$$F_{[1]}(x) = (0x+1)^{n-i} \left(\frac{x}{2} + \frac{1}{2}\right)^{i-1} x, \qquad F_{[2]}(x) = \left(\frac{x}{2} + \frac{1}{2}\right)^{n-i} \cdot (x+0)^{i-1} x.$$

• for k = 2,

$$F_{[1,1]}(x) = (0x+1)^{n-i} \left(\frac{x}{4} + \frac{3}{4}\right)^{i-1} x, \qquad F_{[2,1]}(x) = \left(\frac{x}{4} + \frac{3}{4}\right)^{n-i} \left(\frac{x}{2} + \frac{1}{2}\right)^{i-1} x,$$
$$F_{[1,2]}(x) = \left(\frac{x}{2} + \frac{1}{2}\right)^{n-i} \left(\frac{3x}{4} + \frac{1}{4}\right)^{i-1} x, \qquad F_{[2,2]}(x) = \left(\frac{3x}{4} + \frac{1}{4}\right)^{n-i} (x+0)^{i-1} x.$$

Hooray! $F_L(x)$ has a nice simple pattern

It is utterly important that $F_L(x)$ has a simple pattern from which the underlying structure of the problem can be uncovered. To keep track of the pattern, we define a functional notation:

$$g(a,b) = [ax + (1-a)]^{n-i} \cdot [bx + (1-b)]^{i-1} \cdot x.$$
(9)

With this notation, the function $F_L(x)$ for k = 0, 1 can be expressed as

for k = 0, *F*[](x) = g(0, 1).
for k = 1,

$$F_{[1]}(x) = g\left(0, \frac{1}{2}\right), \ F_{[2]}(x) = g\left(\frac{1}{2}, 1\right).$$

For general $k \ge 1$, a bisection tree in Figure 3 illustrates the connections between the expressions $F_L(x)$ of the k and k + 1 shuffles. At any subtree, the argument (a, b) for the function $g(\cdot)$ of the parent node is bisected, yielding the arguments for the left $(a, \frac{a+b}{2})$ and the right $(\frac{a+b}{2}, b)$ child nodes. It is important to note that the parent's index L_t is expanded by adding the first entry (either 1 or 2) to create the child's indices, i.e. the list indices $[1, L_t]$ and $[2, L_t]$ for the first and second child nodes, respectively.

Let us state and prove this observation.

Theorem 2 (Generating functions). Let $g(a,b) = [ax + (1-a)]^{n-i} \cdot [bx + (1-b)]^{i-1} \cdot x$. For any given $k \ge 1$, and the list indices $L_1, L_2, \ldots, L_{2^k}$,

$$F_{L_t}(x) = g\left(\frac{t-1}{2^k}, \frac{t}{2^k}\right),$$

for $t = 1, 2, ..., 2^k$



Figure 3: A bisection tree structure. Each node of depth k represents $F_{L_t}(x)$ for $t = 1, 2, \ldots, 2^k$.

We use the following lemma to find the sum of the terms in $F_L(x)$ recursively.

Lemma 3. Assume $f(i) = A^{n-i}B^{i-1}x$ where A = ax + (1-a) and B = bx + (1-b). Then,

$$T_1(f) := \sum_{s=1}^{i} \frac{1}{2^{i-1}} \binom{i-1}{s-1} f(s) = A^{n-i} \left(\frac{A+B}{2}\right)^{i-1} x;$$

$$T_2(f) := \sum_{s=i}^{n} \frac{1}{2^{n-i}} \binom{n-i}{s-i} f(s) = \left(\frac{A+B}{2}\right)^{n-i} B^{i-1} x.$$

Proof. First, it is easy to show that

$$\sum_{s=1}^{i} \frac{1}{2^{i-1}} \binom{i-1}{s-1} x^{s-1} = \left(\frac{x}{2} + \frac{1}{2}\right)^{i-1}$$

and

$$\sum_{s=i}^{n} \frac{1}{2^{n-i}} \binom{n-i}{s-i} x^{s-1} = \left(\frac{x}{2} + \frac{1}{2}\right)^{n-i} \cdot (x+0)^{i-1}.$$

For the general case,

$$\sum_{s=1}^{i} \frac{1}{2^{i-1}} \binom{i-1}{s-1} f(s) = \sum_{s=1}^{i} \frac{1}{2^{i-1}} \binom{i-1}{s-1} A^{n-1} \left(\frac{B}{A}\right)^{s-1} x$$
$$= A^{n-1} x \left(\frac{A+B}{2A}\right)^{i-1}, \quad \text{from the above identity}$$
$$= A^{n-i} \left(\frac{A+B}{2}\right)^{i-1} x.$$

This proves the first claim regarding $T_1(f)$. The second claim can be done similarly. \Box

We now return to the proof of Theorem 2.

Proof of Theorem 2. We prove by induction on k = |L|. For the base case, we verify that $F_{[]}(x) = x^i$, is indeed g(0,1). For the induction step, let $L' = [l_{k-1}, ..., l_2, l_1]$ and $L = [l_k, l_{k-1}, ..., l_2, l_1] = [l_k, L']$. Assume $f := F_{L'}(x) = g\left(\frac{t-1}{2^{k-1}}, \frac{t}{2^{k-1}}\right)$. Then it follows from Lemma 3 that $F_L(x) = T_1(f) = g\left(\frac{2t-2}{2^k}, \frac{2t-1}{2^k}\right)$ if $l_k = 1$ and $F_L(x) = T_2(f) =$ $g\left(\frac{2t-1}{2^k}, \frac{2t}{2^k}\right)$ if $l_k = 2$.

Open Problem 1: Find a combinatorial interpretation for the generating function $F_L(x)$ of this card guessing problem.

3 Distribution of p_L when n is large

Exact distribution of p_L

Through the formula of g(a, b) in Theorem 2, it is easy to see that p_L is the sum of two independent binomial random variables and a constant. The following corollary formally states the distribution of p_L , and eventually the exact distribution of $a_{i,j}^{(k)}$.

Corollary 4. For any given $k \ge 1$ and $L_1, L_2, \ldots, L_{2^k}$, let S_{L_t} be a random variable whose probability generating function is given by $F_{L_t}(x)$ defined in Theorem 2. Then,

$$S_{L_t} \sim B_1 + B_2 + 1,$$



where $B_1 \sim Binomial\left(n-i, \frac{t-1}{2^k}\right)$ and $B_2 \sim Binomial\left(i-1, \frac{t}{2^k}\right)$ are independent. In other words, p_{L_t} follows a Poisson Binomial distribution with success probabilities equal to

$$\underbrace{\frac{t-1}{2^k}, \dots, \frac{t-1}{2^k}}_{n-i \ times}, \underbrace{\frac{t}{2^k}, \dots, \frac{t}{2^k}}_{i-1 \ times}, 1.$$

Hence, for fixed values of n, i, k, the distribution $\{a_{i,j}^{(k)} : j = 1, ..., n\}$ is a mixture of 2^k components of Poisson Binomials.

Example. To better visualize the distribution, Figure 4 illustrates the p.m.f. for the case k = 2 shuffles when i = 15 and n = 40.



Figure 4: Probability mass functions $p_{[1,1]}$, $p_{[2,1]}$, $p_{[1,2]}$, $p_{[2,2]}$ for i = 15 and n = 40

In this case, the distribution of each random variable is

$$S_{[1,1]} \sim \operatorname{Binomial}\left(14, \frac{1}{4}\right) + 1, \quad S_{[2,1]} \sim \operatorname{Binomial}\left(25, \frac{1}{4}\right) + \operatorname{Binomial}\left(14, \frac{2}{4}\right) + 1,$$
$$S_{[1,2]} \sim \operatorname{Binomial}\left(25, \frac{2}{4}\right) + \operatorname{Binomial}\left(14, \frac{3}{4}\right) + 1, \quad S_{[2,2]} \sim \operatorname{Binomial}\left(25, \frac{3}{4}\right) + 15.$$

Lyapunov Central Limit Theorem

The Lyapunov CLT generalizes the classical Lindeberg-Feller CLT. The theorem provides a sufficient condition under which the sum of independent (but not necessarily identical) random variables converges in distribution to the standard normal distribution [Durrett (2019)].

Theorem 5 (Lyapunov CLT). Let X_1, X_2, \ldots , be a sequence of independent random variables such that $E(X_n) = \mu_n$ and $Var(X_n) = \sigma_n^2$ are both finite. Define

$$s_n^2 = \sum_{i=1}^n \sigma_i^2.$$

If there exists $\delta > 0$ such that

$$\lim_{n \to \infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^n E\left(|X_i - \mu_i|^{2+\delta}\right) = 0, \qquad (Lyapunov's \ condition)$$

then

$$\frac{1}{s_n} \sum_{i=1}^n E\left(X_i - \mu_i\right) \stackrel{d}{\to} N\left(0, 1\right).$$

The following example gives a sufficient condition for the sum of non-identical Bernoulli random variables to converge to a normal distribution.

Example (Lyapunov's condition for Bernoulli random variables). Let $X_n \sim$ Bernoulli (p_n) be a sequence of independent Bernoulli random variables. Then, $\mu_n = p_n$ and $\sigma_n^2 = p_n (1 - p_n)$. Since

$$E(|X_n - \mu_n|^3) = p_n (1 - p_n)^3 + (1 - p_n) p_n^3 = \sigma_n^2 (1 - 2p_n (1 - p_n)) \le \sigma_n^2$$

it follows that (for $\delta = 1$),

$$\frac{1}{s_n^3} \sum_{i=1}^n E\left(|X_i - \mu_i|^3\right) \le \frac{1}{s_n^3} \sum_{i=1}^n \sigma_n^2 = \frac{1}{s_n}.$$

Thus, if $\lim_{n\to\infty} s_n = \infty$, or equivalently, whenever

$$\sum_{i=1}^{n} p_i \left(1 - p_i \right) \to \infty,$$

the Lyapunov's condition holds for Bernoulli random variables.

Corollary 6. Let 0 < p, q < 1 be fixed, and $T_1 \sim Binomial(m, p)$ and $T_2 \sim Binomial(n, q)$ be two independent binomial random variables. Then,

$$\frac{(T_1 + T_2) - (mp + nq)}{\sqrt{mp(1 - p) + nq(1 - q)}} \stackrel{d}{\to} N(0, 1)$$

whenever $m \to \infty$ or $n \to \infty$ or both hold.

Proof. First note that $T_1 + T_2$ has mean mp + nq and variance mp(1-p) + nq(1-q). The result is immediate as we can write

$$T_1 + T_2 \stackrel{d}{=} \underbrace{X_1 + \dots + X_m}_{X_i \sim \text{Bernoulli}(p)} + \underbrace{X_{m+1} + \dots + X_{m+n}}_{X_i \sim \text{Bernoulli}(q)},$$

and the Lyapunov's condition for Bernoulli random variables,

$$\sum_{i=1}^{m+n} p_i \left(1 - p_i\right) = mp + nq \to \infty,$$

is satisfied whenever $m \to \infty$ or $n \to \infty$ or both hold.

The trivial cases of $p_{[1,1,\dots,1]}$ and $p_{[2,2,\dots,2]}$

Corollaries 4 and 6 tell us that as n is sufficiently large, p_{L_t} will approach a normal distribution, except for the trivial cases that might occur with $p_{[1,1,\dots,1]}$ and $p_{[2,2,\dots,2]}$. In particular, regardless of the magnitude of n, $p_{[1,1,\dots,1]}$ will remain a shifted binomial random variable, $\operatorname{Binomial}\left(i-1,\frac{1}{2^k}\right)+1$, for small i, whereas $p_{[2,2,\dots,2]}$ remains $\operatorname{Binomial}\left(n-i,1-\frac{1}{2^k}\right)+i$, for small n-i.

For the future reference, the modes of $p_{[1,1,\ldots,1]}$ and $p_{[2,2,\ldots,2]}$ are located at

$$j_i = \left\lfloor \frac{i}{2^k} \right\rfloor + 1 \tag{10}$$

and

$$j_i = \left\lfloor (n-i+1)\left(1-\frac{1}{2^k}\right)\right\rfloor + i,\tag{11}$$

respectively, for $i = 1, \ldots, n$.

We shall hereafter call these two cases "the trivial cases". In order not to interrupt the flow of the presentation, we will consider only the non-trivial cases of $p_{[1,1,\dots,1]}$ and $p_{[2,2,\dots,2]}$ (when the CLT applies) in detail, and defer the discussion of the trivial cases to the end of the section.

We immediately obtain the following corollary (for the non-trivial cases).

Corollary 7. Let $k \ge 1$ be fixed, and $t = 1, 2, 3, \ldots, 2^k$. As $n \to \infty$,

$$p_{L_t} \stackrel{d}{\to} N\left(\mu_t, \sigma_t^2\right),$$

where

$$\mu_t = (n-i)\left(\frac{t-1}{2^k}\right) + (i-1)\left(\frac{t}{2^k}\right) + 1$$

and

$$\sigma_t^2 = (n-i)\left(\frac{t-1}{2^k}\right)\left(1 - \frac{t-1}{2^k}\right) + (i-1)\left(\frac{t}{2^k}\right)\left(1 - \frac{t}{2^k}\right)$$



Figure 5: (Figure accompanying Remark 2.) Probability mass functions $p_{[1,1]}$, $p_{[2,1]}$, $p_{[1,2]}$, $p_{[2,2]}$ for i = 15, and n varies from 30 (top row), 50 (middle row), and 100 (bottom row). Observe that regardless of the value of n, the distribution $p_{[1,1]}$ remains the same.

Remark 2. As a practical note, whenever we come across the statement "as $n \to \infty$ " or "large enough n", the condition such as $n \ge N(k)$ can be used for the CLT to be applicable. This can be seen from Figure 5: k = 2 shuffles where we fix i = 15, and vary n from 30, 50, and 100. Here, n = 30 seems pretty large already for the middle two distributions $p_{[2,1]}$ and $p_{[1,2]}$ to converge to a normal distribution. For the case when t = 1, since i = 15 is small, we have the trivial case here. The distribution $p_{[1,1]}$ which is independent of n remains the same shifted Binomial regardless of the values of n (see the first column of the figure). On the other hand, for the case when $t = 2^k$, n - i = n - 15

increases from 15, 35, 85 as n increases from 30, 50, 100, and so the distribution of $p_{[2,2]}$ converges to a normal distribution in this case (see the last column of the figure).

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4 Which peak is highest when n is large?

Our goal now is to compare the heights of the peak of each component p_{L_t} , $t = 1, ..., 2^k$ (for the non-trivial cases). To clearly visualize this, we overlay the plots of $p_{[1,1]}$, $p_{[2,1]}$, $p_{[1,2]}$, $p_{[2,2]}$ on the same axis in Figure 6. Here, we fix n = 100 and vary *i* from 10, 20, 50, and 91. We give an example of i = 91 in order to emphasize the fact that the cases i = 10and i = 100 - 10 + 1 = 91 are mirror images of each other along the vertical direction. The cases when i = 10 and 20 lead to the trivial case of $p_{[1,1]}$.



Figure 6: Probability mass functions $p_{[1,1]}$, $p_{[2,1]}$, $p_{[1,2]}$, $p_{[2,2]}$ for n = 100, and i varies from 10, 20, 50, and 91. Notice that i = 10 and i = 91 are mirror images along the vertical direction.

The smaller the standard deviation, the higher the peak

Recall the standard deviation σ_t of each p_{L_t} given in Corollary 7, and the fact that the component p_{L_t} with smallest standard deviation will have the highest peak. By comparing

the values of σ_t , we obtain the following proposition which can be used to identify the component p_{L^*} which achieves the highest peak among components p_{L_t} , $t = 1, \ldots, 2^k$.

Proposition 8. Let n, i, k be fixed. For $t = 1, ..., 2^k$, the height of the peak of p_{L_t} (when n is large) is approximately equal to $\frac{1}{\sigma_t \sqrt{2\pi}}$, where

$$\sigma_t = \sqrt{\left(n-i\right)\left(\frac{t-1}{2^k}\right)\left(1-\frac{t-1}{2^k}\right) + \left(i-1\right)\left(\frac{t}{2^k}\right)\left(1-\frac{t}{2^k}\right)}.$$
(12)

Denote by p_{L^*} the component whose peak is highest among p_{L_t} , $t = 1, \ldots, 2^k$. Then,

$$p_{L^*} = \begin{cases} p_{[1,1,\dots,1]} & \text{if } i \le \left\lfloor \frac{(n+1)}{2} \right\rfloor \\ p_{[2,2,\dots,2]} & \text{otherwise.} \end{cases}$$
(13)

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Proof. The height $\frac{1}{\sigma_t \sqrt{2\pi}}$ of the peak is due to the bell-shaped curve normal distribution. Using the fact that f(p) := p(1-p) is unimodal and attains a minimum value at the boundary points, it is clear that if $i \leq \left\lfloor \frac{(n+1)}{2} \right\rfloor$, σ_t is smallest when t = 1. Hence, $p_{L^*} = p_{[1,1,\dots,1]}$ in this case. The result of the other case is justified in the same way. \Box

5 Almost non-overlapping $p_{L_1}, p_{L_2}, \ldots, p_{L_{2k}}$

Recall that our ultimate goal is to determine for each i, the index j_i^* that maximizes the mixture distribution

$$a_{i,j}^{(k)} = \frac{1}{2^k} \sum_{t=1}^{2^k} p_{L_t}(j).$$

Thus far, we were able to show that, as n is large, the mixture components p_{L_t} are each normal distribution, and hence $a_{i,j}^{(k)}$ has the Gaussian mixture distribution. We have also identified the component $p_{L_t^*}$ whose peak is highest. Now, since the peaks of any two consecutive components p_{L_t} and $p_{L_{t+1}}$ are $\frac{n-1}{2^k}$ apart (easily verified by taking $\mu_{t+1} - \mu_t$), while the standard deviation of each p_{L_t} is of order $\mathcal{O}(\sqrt{n})$, we can be sure that, as $n \to \infty$, the Gaussian components $p_{L_1}, \ldots, p_{L_{2^k}}$ are hardly overlapping.

Maximum probablity $a_{i,j^*}^{(k)}$ is found!

The fact that the components $p_{L_1}, p_{L_2}, \ldots, p_{L_{2^k}}$ are (almost) non-overlapping (i.e. wellseparated) when n is large is very crucial and tremendously simplifies the problem of locating the maximizer j_i^* of $a_{i,j}^{(k)}$. In particular, the location of the highest peak p_{L^*} , identified previously in Proposition 8, is the maximizer we are seeking.

To see this, Figure 7 (k = 3, i = 15) illustrates the p.m.f of components $p_{L_1}, p_{L_2}, \ldots, p_{L_8}$, and the superposition of the mixture distribution (the red line). We do not assign the weight of 1/8 to each component for the sake of clear visualization of the mixture. The location of the peak of the red line is the maximizer j_i^* . As can be seen from the figure, when n is small (top), the location of $j_i^* = 3$ (red asterisk) does not coincide with the location of the highest peak of p_{L_1} (blue asterisk, at j = 2). On the other hand, as nbecomes larger (bottom), the almost non-overlapping phenomenon occurs: while the peak of p_{L_1} remains at the same height, the peaks of other components $p_{L_t}, t > 1$, get lower and further apart (remember the standard deviation is of order $\mathcal{O}(\sqrt{n})$). Thus, the location of the maximizer j_i^* coincides with the location of the peak of the component p_{L_1} whenever n is sufficiently large.

Remark 3. We actually do not require n to be that close to infinity. The components p_{L_t} can still be overlapping, as long as there is no shift in the location of the maximum peak in the mixture model, and that is all we need. For example, we can see from the middle plot of Figure 7 that we already get the desired outcome when n is as small as 50.

Revisit and settle the trivial cases

Before we end this section, let us now settle the trivial cases. In fact, everything we have discussed so far applies to the trivial cases. In particular the result regarding p_{L^*} in Proposition 8 still holds true. Consider for example the trivial case of $p_{[1,1,\ldots,1]}$ when *i* is small. Being a shifted Binomial with small and constant standard deviation σ_1 , the peak of $p_{[1,1,\ldots,1]}$ will indeed be highest. Figure 7 (when i = 15) also highlights the fact that $p_{[1,1,\ldots,1]}$ does not depend on *n*. In particular, while the peaks of the other p_{L_t} , t > 1 get lower as *n* increases from 30, 50, 400, the peak of $p_{[1,1,\ldots,1]}$ remains at the same height (and same location of course). Proceeding in the same manner, the trivial case of $p_{[2,2,\ldots,2]}$ when n-i is small will achieve the highest peak, the result consistent with the second case of Proposition 8.

6 Game solved!

Now that we have done all the hard work, we are ready to wrap things up. The maximum probablity of $a_{i,j}^{(k)}$ is contributed solely from the peak of $p_{[1,1,\dots,1]}$ if $i \leq \left\lfloor \frac{(n+1)}{2} \right\rfloor$ or $p_{[2,2,\dots,2]}$





Figure 7: Overlay probability mass functions p_{L_1} - p_{L_8} for k = 3 shuffles and i = 15. The red line is the superposition of the mixture distribution (without the weight 1/8.) Top: when n is small, the location of j_i^* does not necessary coincide with the location of the peak of p_{L_1} . Middle and Bottom: when n is large enough, the location of j_i^* coincides with the location of the peak of p_{L_1} .

otherwise. Following immediately from the location of the mode of the shifted Binomial, $p_{[1,1,\ldots,1]}$ and $p_{[2,2,\ldots,2]}$ as given in Equations 10-11 and Proposition 8, the optimal strategy \mathcal{G}^* can now be determined.

Optimal guessing strategy
$$\mathcal{G}^*$$
 for the *i*th position
• If $i \leq \left\lfloor \frac{(n+1)}{2} \right\rfloor$, the best guess is the number $j_i^* = \left\lfloor \frac{i}{2^k} \right\rfloor + 1$;
• If $i \geq \left\lceil \frac{(n+1)}{2} \right\rceil$, the best guess is the number $j_i^* = \left\lfloor (n-i+1)\left(1-\frac{1}{2^k}\right) \right\rfloor + i$.

That is, for a k-time shuffled deck of n cards, one should guess the top half of the deck with sequence

$$\underbrace{1,\ldots,1}_{2^k-1 \text{ times}}, \underbrace{2,\ldots,2}_{2^k \text{ times}}, \underbrace{3,\ldots,3}_{2^k \text{ times}}, \underbrace{4,\ldots,4}_{2^k \text{ times}}, \ldots$$

and guess the bottom half in the reverse manner, i.e.

$$\dots, \underbrace{n-3, \dots, n-3}_{2^k \text{ times}}, \underbrace{n-2, \dots, n-2}_{2^k \text{ times}}, \underbrace{n-1, \dots, n-1}_{2^k \text{ times}}, \underbrace{n, \dots, n}_{2^{k-1} \text{ times}}.$$

Using this strategy, we can calculate the expected number of correct guesses for the whole deck leading to the main result of this paper. It is worth noting that the expected value relies only on the standard deviation of the component $p_{[1,1,\dots,1]}$ of the first half of the deck, as we shall now see.

Theorem 9. For an n-card deck, let $X^{\mathcal{G}^*}$ be the number of correct guesses (according to the optimal strategy \mathcal{G}^*) after k-riffle shuffles. Then,

$$E\left[X^{\mathcal{G}^*}\right] = \frac{2\sqrt{n}}{\sqrt{(2^k - 1)\pi}} + \mathcal{O}(1).$$

Proof. Recall Equation 4: $E\left[X^{\mathcal{G}^*}\right] = \sum_{i=1}^n a_{i,j^*}^{(k)} = \frac{1}{2^k} \sum_{i=1}^n p_L(j_i^*)$, and the standard deviation of $p_{[1,1,\dots,1]}$: $\sigma_1 = \sqrt{\left(i-1\right)\left(\frac{1}{2^k}\right)\left(1-\frac{1}{2^k}\right)}$. By the symmetric property of the probability matrix $P^{(k)}$, we will consider only half of the deck and double the value of the summation. For an asymptotic value n, we have (from Proposition 8):

$$E\left[X^{\mathcal{G}^*}\right] = 2\sum_{i=2}^{\lfloor \frac{(n+1)}{2} \rfloor} \frac{1}{\sqrt{(i-1)(2^k-1)}\sqrt{2\pi}} + \mathcal{O}(1)$$
$$= \frac{2}{\sqrt{(2^k-1)2\pi}} 2\sqrt{\frac{n}{2}} + \mathcal{O}(1)$$
$$= \frac{2\sqrt{n}}{\sqrt{(2^k-1)\pi}} + \mathcal{O}(1).$$

Let us clarify the approximation error terms absorbed in $\mathcal{O}(1)$. Specifically, this includes (i) the error in normal approximations, when *i* is small. The normal distribution gives a less accurate approximation, say, for the first N(k) terms (finite number of terms for a given *k*) of the sum, after which the approximation becomes more and more accurate. The total error in this case is of order $\mathcal{O}(1)$; (ii) the approximation errors due to the CLT (i.e. the normal approximation to the shifted Binomial). These approximation errors are asymptotically negligible compared to that in (i); and finally (iii) the errors of order $\mathcal{O}(1)$ due to approximation of a summation by an integral (in the second equality).

From the theorem, notice that for each additional shuffle, the expected number of correct guesses will go down by roughly a factor of $\sqrt{2}$ (in a huge deck of *n* cards, of course). We also remark that for k = 1, the result boils down to that of [Ciucu (1998)].

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