ON IDENTITIES OF RUGGLES, HORADAM, HOWARD, AND YOUNG

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Abstract. Ruggles (1963) discovered that for integers $n \ge 0$ and $k \ge 1$

$$F_{n+2k} = L_k F_{n+k} + (-1)^{k+1} F_n.$$

Horadam (1965), Howard (2001), and Young (2003) each expanded this identity to generalized linear recurrence relations of orders 2, 3, and integers $r \ge 2$, respectively. In this paper, we let $r \ge 2$ be an integer and w_0, w_1, …, w_{r-1}, and p_1, p_2, …, $p_r \ne 0$ be integers. For $n \ge r$ let

$$w_n = p_1 w_{n-1} + p_2 w_{n-2} + \cdots + p_r w_{n-r}.$$

We will find identities like Ruggles, Horadam, Howard, and Young. These identities will take the form

$$w_{n+rk} = R_k(r-1,r) w_{n+(r-1)k} + R_k(r-2,r) w_{n+(r-2)k} + \cdots + R_k(1,r) w_{n+k} + R_k(0,r) w_n,$$

where, by a result of Young, $R_k(i,r)$ is a linear recurrence relation of order $\binom{r}{i}$ for $i = 0, 1, \ldots, r-1$. Our proof will use the Cayley-Hamilton theorem. Next, we will find the recurrences $R_k(0,r)$ and $R_k(r-1,r)$ in general. Finally, we will explicitly find these identities for orders $r = 3$, $r = 4$ and $r = 5$.

1. Introduction

Let $\{F_n\}$ and $\{L_n\}$ be the Fibonacci and Lucas numbers, respectively. That is, $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$ and $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \ge 2$. Ruggles [4] proved that for integers $n \ge 0$ and $k \ge 1$,

$$F_{n+2k} = L_k F_{n+k} + (-1)^{k+1} F_n.$$

Horadam [1] generalized this result to a general second order recurrence relation.

Theorem 1. *Let w_0, w_1, a, and $b \ne 0$ be integers. Let*

$$w_n = a w_{n-1} + b w_{n-2} \text{ for } n \ge 2.$$

In addition, let $x_0 = 2$, $x_1 = a$, and for $n \ge 2$,

$$x_n = a x_{n-1} + b x_{n-2}.$$

Then for integers $n \ge 0$ and $k \ge 1$,

$$w_{n+2k} = x_k w_{n+k} + (-1)^{k+1} b^k w_n.$$

Howard [2] generalized this result to third order recurrence relations. Young [6] generalized Howard's result for rth order recurrence relations, where $r \ge 2$ is an integer. In this paper, we will let $r \ge 2$ be an integer and let w_0, w_1, …, w_{r-1}, and p_1, p_2, …, $p_r \ne 0$ be integers. For $n \ge r$ let

$$w_n = p_1 w_{n-1} + p_2 w_{n-2} + \cdots + p_r w_{n-r}.$$

We will find identities of the form

$$w_{n+rk} = R_k(r-1,r) w_{n+(r-1)k} + R_k(r-2,r) w_{n+(r-2)k} + \cdots + R_k(1,r) w_{n+k} + R_k(0,r) w_n,$$

where $R_k(i,r)$ is a linear recurrence sequence in k of order $\binom{r}{i}$ for $i = 0, 1, \ldots, r-1$. Our proof will use the Cayley-Hamilton theorem. In addition, we will find the recurrences $R_k(0,r)$ and $R_k(r-1,r)$ in general and we will then try to explicitly find identities of orders $r=3$, $r=4$ and $r=5$.

2. Definition and Lemma

To begin, we need a general definition and a useful lemma.

Definition 1. *Let $r \geq 2$ be an integer. Let w_0, w_1, $\ldots$, w_{r-1}, and p_1, p_2, $\ldots$, $p_r \neq 0$ be integers. Let*

$$w_n = p_1 w_{n-1} + p_2 w_{n-2} + \cdots + p_r w_{n-r} \text{ for } n \geq r.$$

We now state our lemma.

Lemma 1. *Let $k \geq 1$ be an integer. Let M be the $r \times r$ matrix given by*

$$\begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_{r-1} & p_r \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Let

$$p(x) = \det(xI - M^k) = \sum_{i=0}^{r} C_k(i,r) x^i$$

be the characteristic polynomial of M^k. Then

$$\sum_{i=0}^{r} C_k(i,r) w_{n+ik} = 0.$$

Proof. By the Cayley-Hamilton Theorem, every matrix satisfies its characteristic polynomial. Therefore,

$$p(M^k) = \det(M^k I - M^k) = \sum_{i=0}^{r} C_k(i,r)(M^k)^i = 0. \tag{1}$$

Multiplying both sides of (1) on the right by

$$\begin{pmatrix} w_n \\ w_{n-1} \\ \vdots \\ w_{n-r+1} \end{pmatrix}$$

gives

$$\sum_{i=0}^{r} C_k(i,r) M^{ik} \begin{pmatrix} w_n \\ w_{n-1} \\ \vdots \\ w_{n-r+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2}$$

But, it can be shown, by induction on m, that

$$\begin{pmatrix} p_1 & p_2 & p_3 & \cdots & p_{r-1} & p_r \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^m \begin{pmatrix} w_n \\ w_{n-1} \\ \vdots \\ w_{n-r+1} \end{pmatrix} = \begin{pmatrix} w_{n+m} \\ w_{n+m-1} \\ \vdots \\ w_{n+m-r+1} \end{pmatrix}. \tag{3}$$

Letting $m = ik$ in (3) and substituting the right-hand side of (3) into (2), we obtain

$$\sum_{i=0}^{r} C_k(i,r) \begin{pmatrix} w_{n+ik} \\ w_{n+ik-1} \\ \vdots \\ w_{n+ik-r+1} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{r} C_k(i,r) w_{n+ik} \\ \sum_{i=0}^{r} C_k(i,r) w_{n+ik-1} \\ \vdots \\ \sum_{i=0}^{r} C_k(i,r) w_{n+ik-r+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4}$$

Equating the first component of the two column vectors of (4) gives the result. □

Since the leading coefficient of the characteristic polynomial of M^k is 1, we have $C_k(r,r) = 1$. Therefore, we can rewrite the identity in the lemma as

$$w_{n+rk} = -C_k(r-1,r) w_{n+(r-1)k} - C_k(r-2,r) w_{n+(r-2)k} - \cdots - C_k(0,r) w_n.$$

We next let $R_k(i,r) = -C_k(i,r)$ for $i = 0, 1, \ldots, r-1$. Therefore, our identity will take the form

$$w_{n+rk} = R_k(r-1,r) w_{n+(r-1)k} + R_k(r-2,r) w_{n+(r-2)k} + \cdots + R_k(0,r) w_n.$$

First, we will find this identity for the Tribonacci sequence. Then, we will determine, in general, the sequence $R_k(r-1,r)$ and $R_k(0,r)$. Finally, we will investigate the cases $r = 3$, $r = 4$, and $r = 5$. Therefore, we will try to determine the sequences $R_k(1,3)$; $R_k(1,4)$, $R_k(2,4)$; and $R_k(1,5)$, $R_k(2,5)$, and $R_k(3,5)$. Young [6] proved that each sequence $R_k(i,r)$ is a recurrence relation of order $\binom{r}{i}$. Using this fact and a computer algebra system, we will find each of these recurrence relations.

3. Howard's Identity for the Tribonacci Sequence

To demonstrate the use of Lemma 1, we will find an identity for the Tribonacci sequence. In this example, we let $r = 3$ and $p_1 = p_2 = p_3 = 1$.

Definition 2. [3, A000073] *Let $T_0 = 0$, $T_1 = 0$, and $T_2 = 1$. Let*

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \text{ for } n \geq 3.$$

The polynomials producing $R_k(2,3)$, $R_k(1,3)$, and $R_k(0,3)$ are the following.

$$\det(xI - I) = \det \begin{pmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{pmatrix} = x^3 - 3x^2 + 3x - 1.$$

$$\det(xI - M) = \det \begin{pmatrix} x-1 & -1 & -1 \\ -1 & x & 0 \\ 0 & -1 & x \end{pmatrix} = x^3 - x^2 - x - 1.$$

$$\det(xI - M^2) = \det \begin{pmatrix} x-2 & -2 & -1 \\ -1 & x-1 & -1 \\ -1 & 0 & x \end{pmatrix} = x^3 - 3x^2 - x - 1.$$

$$\det(xI - M^3) = \det \begin{pmatrix} x-4 & -3 & -2 \\ -2 & x-2 & -1 \\ -1 & -1 & x-1 \end{pmatrix} = x^3 - 7x^2 + 5x - 1.$$

$$\det(xI - M^4) = \det \begin{pmatrix} x-7 & -6 & -4 \\ -4 & x-3 & -2 \\ -2 & -2 & x-1 \end{pmatrix} = x^3 - 11x^2 - 5x - 1.$$

$$\det(xI - M^5) = \det \begin{pmatrix} x-13 & -11 & -7 \\ -7 & x-6 & -4 \\ -4 & -3 & x-2 \end{pmatrix} = x^3 - 21x^2 - x - 1.$$

$$\det(xI - M^6) = \det \begin{pmatrix} x-24 & -20 & -13 \\ -13 & x-11 & -7 \\ -7 & -6 & x-4 \end{pmatrix} = x^3 - 39x^2 + 11x - 1.$$

Here are the beginning values of the sequences.

Table 1. Values of Specific Third Order Sequences

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
T_k	0	0	1	1	2	4	7	13	24	44	81	149	274	504	927	1705
$R_k(2,3)$	3	1	3	7	11	21	39	71	131	241	443	815	1499	2757	5071	9327
$R_k(1,3)$	-3	1	1	-5	5	1	-11	15	-3	-23	41	-21	-43	105	-83	-65
$R_k(0,3)$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Definition 3. [3, A001644] *Let $a_0 = 3$, $a_1 = 1$, and $a_2 = 3$. Let*

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \text{ for } n \geq 3.$$

Definition 4. [3, A073145] *Let $b_0 = -3$, $b_1 = 1$, and $b_2 = 1$. Let*

$$b_n = -b_{n-1} - b_{n-2} + b_{n-3} \text{ for } n \geq 3.$$

We now have the following theorem.

Theorem 2. *Let $n \geq 0$ and $k \geq 1$. Then*

$$T_{n+3k} = a_k T_{n+2k} + b_k T_{n+k} + T_n.$$

4. The Recurrence $R_k(r-1, r)$

In this section, we will determine the sequence $R_k(r-1, r)$ in general.

Definition 5. *Let $r \geq 2$ be a positive integer and let $p_1, p_2, \ldots, p_r \neq 0$ be integers. Let $a_0 = 0$, $a_1 = 0$, $\ldots$, $a_{r-2} = 0$, and $a_{r-1} = 1$. Let*

$$a_n = p_1 a_{n-1} + p_2 a_{n-2} + \cdots + p_r a_{n-r}$$

for $n \geq r$.

We begin with a lemma.

Lemma 2. *Let k be a positive integer. Then*

$$M^k =$$
$$\begin{pmatrix} a_{k+r-1} & p_2a_{k+r-2}+p_3a_{k+r-3}+\cdots+p_ra_k & p_3a_{k+r-2}+\cdots+p_ra_{k+1} & \cdots & p_ra_{k+r-2} \\ a_{k+r-2} & p_2a_{k+r-3}+p_3a_{k+r-4}+\cdots+p_ra_{k-1} & p_3a_{k+r-3}+\cdots+p_ra_k & \cdots & p_ra_{k+r-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_k & p_2a_{k-1}+p_3a_{k-2}+\cdots+p_ra_{k-r+1} & p_3a_{k-1}+\cdots+p_ra_{k-r-2} & \cdots & p_ra_{k-1} \end{pmatrix}.$$

Proof. The proof of the lemma is by induction on k. □

Therefore, for a positive integer k, the characteristic polynomial of M^k is

$$\det(xI - M^k) =$$
$$\det\begin{pmatrix} x-a_{k+r-1} & -p_2a_{k+r-2}-p_3a_{k+r-3}-\cdots-p_ra_k & \cdots & -p_ra_{k+r-2} \\ -a_{k+r-2} & x-p_2a_{k+r-3}-p_3a_{k+r-4}-\cdots-p_ra_{k-1} & \cdots & -p_ra_{k+r-3} \\ \cdots & \cdots & \cdots & \cdots \\ -a_k & -p_2a_{k-1}-p_3a_{k-2}-\cdots-p_ra_{k-r+1} & \cdots & x-p_ra_{k-1} \end{pmatrix}.$$

By examining the $-x^{r-1}$ term of the determinant we observe that the sequence $R_k(r-1,r)$ is

$$\begin{aligned} &a_{k+r-1} + (p_2a_{k+r-3}+\cdots+p_ra_{k-1}) + (p_3a_{k+r-4}+\cdots+p_ra_{k-1}) + \cdots + p_ra_{k-1} \\ &= a_{k+r-1} + p_2a_{k+r-3} + 2p_3a_{k+r-4} + \cdots + (r-1)p_ra_{k-1}, \end{aligned}$$

where k is a positive integer.

To make the notation easier to write, we make the following definition.

Definition 6. *Let $x_0 = r$ and for any positive integer k, let*

$$x_k = a_{k+r-1} + p_2a_{k+r-3} + 2p_3a_{k+r-4} + \cdots + (r-1)p_ra_{k-1}.$$

The following theorem shows that x_k is a linear recurrence of order r and gives its recurrence.

Theorem 3. *Let $n \geq r+1$ be an integer. Then*

$$x_n = p_1x_{n-1} + p_2x_{n-2} + \cdots + p_rx_{n-r}.$$

Proof. Let $n \geq r+1$ be an integer. From the definition of the sequence $\{x_k\}$, for $k = n-1, \ldots, n-r$ we have that

$$x_k = a_{k+r-1} + p_2a_{k+r-3} + 2p_3a_{k+r-4} + \cdots + (r-1)p_ra_{k-1}. \tag{5}$$

For $k = n-1, \ldots, n-r$, multiply the right-hand side of (5) by $p_1, p_2, \ldots, p_r$, respectively. Adding the first terms of each of the r expressions, we have

$$p_1a_{n+r-2} + p_2a_{n+r-3} + \cdots + p_ra_{n-1} = a_{n+r-1}.$$

Adding the second terms of each of the r expressions, we have

$$p_2(p_1a_{n+r-4} + p_2a_{n+r-5} + \cdots + p_ra_{n-3} = p_2a_{n+r-3}.$$

Adding the third terms of each of the r expressions, we have

$$2p_3(p_1a_{n+r-5} + p_2a_{n+r-6} + \cdots + p_ra_{n-4} = 2p_3a_{n+r-4}.$$

Continue this process to include the rth term.

The final result is

$$a_{n+r-1} + p_2 a_{n+r-3} + \cdots + (r-1) p_r a_n = x_n.$$

This is what we wanted to prove. □

5. The Recurrence $R_k(0, r)$

In this section, we will determine the sequence $R_k(0, r)$ in general. We will prove the following theorem.

Theorem 4. *Let k be a non-negative integer. Then*

$$R_k(0, r) = \begin{cases} p_r^k, & \text{if } r \text{ is odd;} \\ (-1)^{k+1} p_r^k, & \text{if } r \text{ is even.} \end{cases}$$

To obtain the recurrence $R_k(0, r)$, we evaluate $\det(xI - M^k)$ at $x = 0$. In general, this sequence is

$$\det \begin{pmatrix} -a_{k+r-1} & -p_2 a_{k+r-2} - p_3 a_{k+r-3} - \cdots - p_r a_k & -p_3 a_{k+r-2} - \cdots - p_r a_{k+1} & \cdots & -p_r a_{k+r-2} \\ -a_{k+r-2} & -p_2 a_{k+r-3} - p_3 a_{k+r-4} - \cdots - p_r a_{k-1} & -p_3 a_{k+r-3} - \cdots - p_r a_k & \cdots & -p_r a_{k+r-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_k & -p_2 a_{k-1} - p_3 a_{k-2} - \cdots - p_r a_{k-r+1} & -p_3 a_{k-1} - \cdots - p_r a_{k-r-2} & \cdots & -p_r a_{k-1} \end{pmatrix} \tag{6}$$

To continue the computation of (6), we need the following lemma from Turnbull [5, p. 31].

Lemma 3. *Let $r \geq 2$ be an integer. An $r \times r$ determinant is unaltered in value by adding to one of its columns any linear combination of its other columns.*

Now we will compute the determinant in (6) with the help of two lemmas.

Lemma 4. *Let k be a positive integer. Then*

$$\det \begin{pmatrix} -a_{k+r-1} & -p_2 a_{k+r-2} - p_3 a_{k+r-3} - \cdots - p_r a_k & -p_3 a_{k+r-2} - \cdots - p_r a_{k+1} & \cdots & -p_r a_{k+r-2} \\ -a_{k+r-2} & -p_2 a_{k+r-3} - p_3 a_{k+r-4} - \cdots - p_r a_{k-1} & -p_3 a_{k+r-3} - \cdots - p_r a_k & \cdots & -p_r a_{k+r-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_k & -p_2 a_{k-1} - p_3 a_{k-2} - \cdots - p_r a_{k-r+1} & -p_3 a_{k-1} - \cdots - p_r a_{k-r-2} & \cdots & -p_r a_{k-1} \end{pmatrix}$$

$$= -p_r^{r-1} \det \begin{pmatrix} a_k & a_{k+1} & \cdots & a_{k+r-1} \\ a_{k-1} & a_k & \cdots & a_{k+r-2} \\ \cdots & \cdots & \cdots & \\ a_{k-r+1} & a_{k-r+2} & \cdots & a_k \end{pmatrix}.$$

Proof. First of all, we factor (-1) from every column of the matrix. Therefore, our initial determinant is equal to

$$(-1)^r \det \begin{pmatrix} a_{k+r-1} & p_2 a_{k+r-2} + p_3 a_{k+r-3} + \cdots + p_r a_k & p_3 a_{k+r-2} + \cdots + p_r a_{k+1} & \cdots & p_r a_{k+r-2} \\ a_{k+r-2} & p_2 a_{k+r-3} + p_3 a_{k+r-4} + \cdots + p_r a_{k-1} & p_3 a_{k+r-3} + \cdots + p_r a_k & \cdots & p_r a_{k+r-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_k & p_2 a_{k-1} + p_3 a_{k-2} + \cdots + p_r a_{k-r+1} & p_3 a_{k-1} + \cdots + p_r a_{k-r-2} & \cdots & p_r a_{k-1} \end{pmatrix}. \tag{7}$$

We now start with the determinant

$$\det \begin{pmatrix} a_k & a_{k+1} & \cdots & a_{k+r-1} \\ a_{k-1} & a_k & \cdots & a_{k+r-2} \\ \cdots & \cdots & \cdots & \\ a_{k-r+1} & a_{k-r+2} & \cdots & a_k \end{pmatrix}$$

and work out way backwards to (7).

We first replace the first column by p_r times the first column plus p_{r-1} times the second column, plus $\cdots$ plus p_2 times the next to last column. Next, we replace the second column by p_r times the second column plus $\cdots$ plus p_3 times the next to last column. Continuing this process, we finally replace the next to last column by p_r times the next to last column. We know how the determinant changes by Lemma 3.

Once we have this new matrix, we swap columns r and $r-1$, then columns $r-1$ and $r-2$. We continue this process until we finally swap columns 2 and 1.

Counting the number of swaps and number of times we multiplied by p_r, we have the result. $\square$

To continue the proof we need the following lemma.

Lemma 5. *Let $r \geq 2$ be an integer. Then for $k \geq r-1$,*

$$\det \begin{pmatrix} a_k & a_{k+1} & \cdots & a_{k+r-1} \\ a_{k-1} & a_k & \cdots & a_{k+r-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k-r+1} & a_{k-r+2} & \cdots & a_k \end{pmatrix} = \begin{cases} p_r^{k-r+1}, & \text{if } r \text{ is odd;} \\ (-1)^{k+1} p_r^{k-r+1}, & \text{if } r \text{ is even.} \end{cases}$$

Proof. The proof of the lemma will be by induction on k. For $k = r-1$, we have

$$\det \begin{pmatrix} a_{r-1} & a_r & \cdots & a_{2r-2} \\ a_{r-2} & a_{r-1} & \cdots & a_{2r-3} \\ \cdots & \cdots & \cdots & \cdots \\ a_0 & a_1 & \cdots & a_{r-1} \end{pmatrix} = 1$$

so the base step is true.

Next, assume the result is true for some $k-1 \geq r-1$ and attempt to prove the result is true for k. We start with the determinant

$$\det \begin{pmatrix} a_k & a_{k+1} & \cdots & a_{k+r-1} \\ a_{k-1} & a_k & \cdots & a_{k+r-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k-r+1} & a_{k-r+2} & \cdots & a_k \end{pmatrix}.$$

We next replace the last column entries by their recurrence relation. This is the determinant

$$\det \begin{pmatrix} a_k & a_{k+1} & \cdots & p_1 a_{k+r-2} + p_2 a_{k+r-3} + \cdots + p_r a_{k-1} \\ a_{k-1} & a_k & \cdots & p_1 a_{k+r-3} + p_2 a_{k+r-4} + \cdots + p_r a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k-r+1} & a_{k-r+2} & \cdots & p_1 a_{k+1} + p_2 a_k + \cdots + p_r a_{k-r} \end{pmatrix}.$$

Applying Lemma 3, ignoring the $p_1, \ldots, p_{r-1}$, considering p_r, and swapping columns r and $r-1, \ldots,$ columns 2 and 1, we have the resulting determinant is

$$p_r(-1)^{r-1} \det \begin{pmatrix} a_{k-1} & a_k & \cdots & a_{k+r-3} \\ a_{k-2} & a_{k-1} & \cdots & a_{k+r-4} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k-r} & a_{k-r+1} & \cdots & a_{k-1} \end{pmatrix}.$$

Now in either case, if r is odd or r is even, the result is true for k. Therefore, by the principle of mathematical induction, the result is true for all $k \geq r-1$. $\square$

Putting both of these lemmas together and using the fact the $R_k(0,r)$ is the coefficient of $-x^0 = -1$, we drop the minus sign to give the result

Therefore, the sequence $R_k(0,r)$ is

$$R_k(0,r) = p_r^{r-1} \cdot \begin{cases} p_r^{k-r+1}, & \text{if } r \text{ is odd;} \\ (-1)^{k+1} p_r^{k-r+1}, & \text{if } r \text{ is even.} \end{cases} = \begin{cases} p_r^{k}, & \text{if } r \text{ is odd;} \\ (-1)^{k+1} p_r^{k}, & \text{if } r \text{ is even.} \end{cases}$$

This is the statement of the theorem.

6. An Explicit Formula for Howard's Third Order Recurrence

We next state the definitions we need to find an explicit formula for Howard's third order result.

Definition 7. *Let w_0, w_1, w_2, a, b, and $c \neq 0$ be integers. Let*

$$w_n = aw_{n-1} + bw_{n-2} + cw_{n-3} \text{ for } n \geq 3.$$

Using Lemma 1, Young's result, and a computer algebra system, we can calculate the sequences $R_k(2,3)$, $R_k(1,3)$, and $R_k(0,3)$. This leads to the following definitions and theorem.

Definition 8. *Let $x_0 = 3$, $x_1 = a$, and $x_2 = a^2 + 2b$. Let*

$$x_n = ax_{n-1} + bx_{n-2} + cx_{n-3} \text{ for } n \geq 3.$$

Definition 9. *Let $y_0 = -3$, $y_1 = b$, and $y_2 = 2ac - b^2$. Let*

$$y_n = -by_{n-1} - acy_{n-2} + c^2 y_{n-3} \text{ for } n \geq 3,$$

Theorem 5. *Let $n \geq 0$ and $k \geq 1$ be integers. Then*

$$w_{n+3k} = x_k w_{n+2k} + y_k w_{n+k} + c^k w_n.$$

7. An Explicit Formula for Young's Fourth Order Result

We next state the definitions we need to find an explicit formula for Young's fourth order result.

Definition 10. *Let w_0, w_1, w_2, w_3, a, b, c, and $d \neq 0$ be integers. Let*

$$w_n = aw_{n-1} + bw_{n-2} + cw_{n-3} + dw_{n-4} \text{ for } n \geq 4.$$

Again, using Lemma 1, Young's result, and a computation using a computer algebra system, we can calculate the sequences $R_k(3,4)$, $R_k(2,4)$, $R_k(1,4)$, and $R_k(0,4)$. This leads to the following definitions and theorem.

Definition 11. *Let $x_0 = 4$, $x_1 = a$, $x_2 = a^2 + 2b$, and $x_3 = a^3 + 3ab + 3c$. Let*

$$x_n = ax_{n-1} + bx_{n-2} + cx_{n-3} + dx_{n-4} \text{ for } n \geq 4.$$

Definition 12. *Let $y_0 = -6$, $y_1 = b$, $y_2 = 2ac - b^2 + 2d$, $y_3 = 3a^2d + b^3 + 3bd - 3abc - 3c^2$, $y_4 = -4a^2bd - 2a^2c^2 + 4ab^2c - 8acd - b^4 - 4b^2d + 4bc^2 - 6d^2$, and $y_5 = -5a^3cd + 5a^2b^2d + 5a^2bc^2 - 5a^2d^2 - 5ab^3c + 5abcd + 5ac^3 + b^5 + 5b^3d - 5b^2c^2 + 5bd^2 + 5c^2d$. Let*

$$\begin{aligned} y_n = {} & -by_{n-1} - (d + ac)y_{n-2} + (c^2 - 2bd - a^2d)y_{n-3} \\ & + d(d + ac)y_{n-4} - bd^2 y_{n-5} + d^3 y_{n-6} \text{ for } n \geq 6. \end{aligned}$$

Definition 13. *Let $z_0 = 4$, $z_1 = c$, $z_2 = c^2 - 2bd$, and $z_3 = 3ad^2 + c^3 - 3bcd$. Let*

$$z_n = cz_{n-1} - bdz_{n-2} + ad^2 z_{n-3} + d^3 z_{n-4} \text{ for } n \geq 4.$$

Theorem 6. *Let $n \geq 0$ and $k \geq 1$. Then*

$$w_{n+4k} = x_k w_{n+3k} + y_k w_{n+2k} + z_k w_{n+k} + (-1)^{k+1} d^k w_n.$$

8. An Explicit Formula for Young's Fifth Order Result

We next state the definitions we need to find an explicit formula for Young's fourth order result.

Definition 14. *Let w_0, w_1, w_2, w_3, w_4, a, b, c, d, and $e \neq 0$ be integers. Let*

$$w_n = aw_{n-1} + bw_{n-2} + cw_{n-3} + dw_{n-4} + ew_{n-5} \text{ for } n \geq 5.$$

Again, using Lemma 1, Young's result, and an extensive computation using a computer algebra system, we can calculate the sequences $R_k(4,5)$, $R_k(3,5)$, $R_k(2,5)$, $R_k(1,5)$, and $R_k(0,5)$. This leads to the following definitions and theorem. The calculations and sequences can be found in Appendix I. With the definitions in Appendix I, we have the following result.

Theorem 7. *Let $n \geq 0$ and $k \geq 1$. Then*

$$w_{n+5k} = x_k w_{n+4k} + y_k w_{n+3k} + z_k w_{n+2k} + v_k w_{n+k} + e^k w_n.$$

9. Appendix I

Let a, b, c, d, and $e \neq 0$ be integers. Let M be the 5×5 matrix

$$\begin{pmatrix} a & b & c & d & e \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let x_k, y_k, z_k and v_k be the coefficient of $-x^4$, $-x^3$, $-x^2$, and $-x^1$ in the $\det(xI - M^k)$, respectively. Compute the first 10 terms of each sequence using a computer algebra system.

```
##################################################
det(x*I-I)
= x^5 - 5*x^4 + 10*x^3 - 10*x^2 + 5*x - 1

##################################################
det(x*I-M)
= x^5 - a*x^4 - b*x^3 - c*x^2 - d*x - e

##################################################
det(x*I-M^2)
= x^5 + (-a^2 - 2*b)*x^4 + (-2*c*a + (b^2 - 2*d))*x^3 + (-2*e*a + (2*d*b - c^
2))*x^2 + (-2*e*c + d^2)*x - e^2

##################################################
det(x*I-M^3)
= x^5 + (-a^3 - 3*b*a - 3*c)*x^4 + (-3*d*a^2 + (3*c*b - 3*e)*a + (-b^3 - 3*d*
b + 3*c^2))*x^3 + ((3*e*c - 3*d^2)*a + (-3*e*b^2 + 3*d*c*b + (-c^3 - 3*e*d))*x^
2 + (-3*e^2*b + (3*e*d*c - d^3))*x - e^3
```

```
##################################################
det(x*I-M^4)
= x^5 + (-a^4 - 4*b*a^2 - 4*c*a + (-2*b^2 - 4*d))*x^4 + (-4*e*a^3 + (4*d*b +
2*c^2)*a^2 + (-4*c*b^2 - 8*e*b + 8*d*c)*a + (b^4 + 4*d*b^2 - 4*c^2*b + (-4*e*c +
 6*d^2)))*x^3 + (-6*e^2*a^2 + (8*e*d*b + (4*e*c^2 - 4*d^2*c))*a + ((-4*e*c - 2*d
^2)*b^2 + (4*d*c^2 - 4*e^2)*b + (-c^4 + 8*e*d*c - 4*d^3)))*x^2 + (-4*e^3*a + (4*
e^2*d*b + (2*e^2*c^2 - 4*e*d^2*c + d^4)))*x - e^4

##################################################
det(x*I-M^5)
= x^5 + (-a^5 - 5*b*a^3 - 5*c*a^2 + (-5*b^2 - 5*d)*a + (-5*c*b - 5*e))*x^4 +
 ((5*e*b + 5*d*c)*a^3 + (-5*d*b^2 - 5*c^2*b + (10*e*c + 5*d^2))*a^2 + (5*c*b^3 +
 10*e*b^2 - 5*d*c*b + (-5*c^3 + 15*e*d))*a + (-b^5 - 5*d*b^3 + 5*c^2*b^2 + (15*e
*c - 5*d^2)*b + (-5*d*c^2 + 10*e^2)))*x^3 + ((-5*e^2*c - 5*e*d^2)*a^2 + (-5*e^2*
b^2 + (5*e*d*c + 5*d^3)*b + (5*e*c^3 - 5*d^2*c^2 - 15*e^2*d))*a + (5*e*d*b^3 + (
-5*e*c^2 - 5*d^2*c)*b^2 + (5*d*c^3 - 15*e^2*c + 10*e*d^2)*b + (-c^5 + 10*e*d*c^2
 - 5*d^3*c - 10*e^3)))*x^2 + (5*e^3*d*a + ((5*e^3*c - 5*e^2*d^2)*b + (-5*e^2*d*c
^2 + 5*e*d^3*c + (-d^5 + 5*e^4))))*x - e^5

##################################################
det(x*I-M^6)
= x^5 + (-a^6 - 6*b*a^4 - 6*c*a^3 + (-9*b^2 - 6*d)*a^2 + (-12*c*b - 6*e)*a +
 (-2*b^3 - 6*d*b - 3*c^2))*x^4 + ((6*e*c + 3*d^2)*a^4 + (-6*e*b^2 - 12*d*c*b + (
-2*c^3 + 12*e*d))*a^3 + (6*d*b^3 + 9*c^2*b^2 + (-18*d*c^2 + 9*e^2))*a^2 + (-6*c*
b^4 - 12*e*b^3 + 12*c^3*b - 18*d^2*c)*a + (b^6 + 6*d*b^4 - 6*c^2*b^3 + (-18*e*c
+ 9*d^2)*b^2 - 6*e^2*b + (3*c^4 - 12*e*d*c - 2*d^3)))*x^3 + (-2*e^3*a^3 + (18*e^
2*d*b + (-9*e^2*c^2 - 3*d^4))*a^2 + (-18*e*d^2*b^2 + (12*d^3*c + 12*e^3)*b + (6*
e*c^4 - 6*d^2*c^3 - 12*e*d^3))*a + (-3*e^2*b^4 + (12*e*d*c + 2*d^3)*b^3 + (-6*e*
c^3 - 9*d^2*c^2)*b^2 + (6*d*c^4 - 18*e^2*c^2 + 6*d^4)*b + (-c^6 + 12*e*d*c^3 - 6
*d^3*c^2 - 6*e^3*c - 9*e^2*d^2)))*x^2 + ((6*e^4*c - 6*e^3*d^2)*a + (3*e^4*b^2 +
(-12*e^3*d*c + 6*e^2*d^3)*b + (-2*e^3*c^3 + 9*e^2*d^2*c^2 - 6*e*d^4*c + (d^6 - 6
*e^4*d))))*x - e^6

##################################################
det(x*I-M^7)
= x^5 + (-a^7 - 7*b*a^5 - 7*c*a^4 + (-14*b^2 - 7*d)*a^3 + (-21*c*b - 7*e)*a^
2 + (-7*b^3 - 14*d*b - 7*c^2)*a + (-7*c*b^2 - 7*e*b - 7*d*c))*x^4 + (7*e*d*a^5 +
 ((-14*e*c - 7*d^2)*b + (-7*d*c^2 + 7*e^2))*a^4 + (7*e*b^3 + 21*d*c*b^2 + (7*c^3
 + 7*e*d)*b + (-21*e*c^2 - 21*d^2*c))*a^3 + (-7*d*b^4 - 14*c^2*b^3 + (-14*e*c -
7*d^2)*b^2 + (35*d*c^2 + 7*e^2)*b + (7*c^4 - 35*e*d*c - 14*d^3))*a^2 + (7*c*b^5
+ 14*e*b^4 + 7*d*c*b^3 + (-21*c^3 + 14*e*d)*b^2 + (-35*e*c^2 + 14*d^2*c)*b + (21
*d*c^3 - 21*e^2*c - 21*e*d^2))*a + (-b^7 - 7*d*b^5 + 7*c^2*b^4 + (21*e*c - 14*d^
2)d*b^3 + (7*d*c^2 + 14*e^2)*b^2 + (-7*c^4 + 7*e*d*c - 7*d^3)*b + (-7*e*c^3 + 14*
d^2*c^2 - 7*e^2*d)))*x^3 + ((7*e^3*c - 14*e^2*d^2)*a^3 + (-14*e^3*b^2 + (14*e^2*
d*c + 21*e*d^3)*b + (-14*e^2*c^3 + 7*e*d^2*c^2 - 7*d^4*c - 21*e^3*d))*a^2 + (21*
e^2*d*b^3 + (7*e^2*c^2 - 35*e*d^2*c - 7*d^4)*b^2 + (-7*e*d*c^3 + 21*d^3*c^2 + 7*
e^3*c + 35*e^2*d^2)*b + (7*e*c^5 - 7*d^2*c^4 - 14*e^2*d*c^2 + 7*e*d^3*c + (-7*d^
```

```
5 - 7*e^4)))*a + ((-7*e^2*c - 7*e*d^2)*b^4 + (21*e*d*c^2 + 7*d^3*c - 7*e^3)*b^3
+ (-7*e*c^4 - 14*d^2*c^3 + 35*e^2*d*c - 21*e*d^3)*b^2 + (7*d*c^5 - 21*e^2*c^3 -
14*e*d^2*c^2 + 14*d^4*c + 21*e^3*d)*b + (-c^7 + 14*e*d*c^4 - 7*d^3*c^3 - 14*e^3*
c^2 + 7*e^2*d^2*c - 7*e*d^4)))*x^2 + ((7*e^5*b + (-14*e^4*d*c + 7*e^3*d^3))*a +
(-7*e^4*d*b^2 + (-7*e^4*c^2 + 21*e^3*d^2*c - 7*e^2*d^4)*b + (7*e^3*d*c^3 - 14*e^
2*d^3*c^2 + (7*e*d^5 - 7*e^5)*c + (-d^7 + 7*e^4*d^2))))*x - e^7

##################################################
det(x*I-M^8)
= x^5+(-a^8-8*b*a^6-8*c*a^5+(-20*b^2-8*d)*a^4+(-32*c*b-8*e)*a^3+(-16*b^3-24*d*b-12
*c^2)*a^2+(-24*c*b^2-16*e*b-16*d*c)*a+(-2*b^4-8*d*b^2-8*c^2*b+(-8*e*c-4*d^2)))*x
^4+(4*e^2*a^6+(-16*e*d*b+(-8*e*c^2-8*d^2*c))*a^5+((24*e*c+12*d^2)*b^2+(24*d*c^2+
8*e^2)*b+(2*c^4-48*e*d*c-8*d^3))*a^4+(-8*e*b^4-32*d*c*b^3+(-16*c^3-32*e*d)*b^2+(
32*e*c^2+32*d^2*c)*b+(32*d*c^3-16*e^2*c-48*e*d^2))*a^3+(8*d*b^5+20*c^2*b^4+(32*e
*c+16*d^2)*b^3+(-48*d*c^2+8*e^2)*b^2+(-24*c^4-32*e*d*c+16*d^3)*b+(16*e*c^3+56*d^
2*c^2-48*e^2*d))*a^2+(-8*c*b^6-16*e*b^5-16*d*c*b^4+(32*c^3-32*e*d)*b^3+(80*e*c^2
-16*d^2*c)*b^2+(-32*d*c^3+32*e^2*c-32*e*d^2)*b+(-8*c^5+32*e*d*c^2+32*d^3*c-8*e^3
))*a+(b^8+8*d*b^6-8*c^2*b^5+(-24*e*c+20*d^2)*b^4+(-16*d*c^2-16*e^2)*b^3+(12*c^4-
32*e*d*c+16*d^3)*b^2+(32*e*c^3-16*d^2*c^2-24*e^2*d)*b+(-8*d*c^4+20*e^2*c^2+8*e*d
^2*c+6*d^4)))*x^3+(-6*e^4*a^4+(32*e^3*d*b+(16*e^3*c^2-16*e^2*d^2*c-8*e*d^4))*a^3
+((-16*e^3*c-56*e^2*d^2)*b^2+(16*e^2*d*c^2+32*e*d^3*c+(8*d^5+8*e^4))*b+(-20*e^2*
c^4+16*e*d^2*c^3-12*d^4*c^2+32*e^3*d*c-48*e^2*d^3))*a^2+(-8*e^3*b^4+(32*e^2*d*c+
32*e*d^3)*b^3+(16*e^2*c^3-48*e*d^2*c^2-24*d^4*c-32*e^3*d)*b^2+(-16*e*d*c^4+32*d^
3*c^3+32*e^3*c^2-32*e^2*d^2*c+48*e*d^4)*b+(8*e*c^6-8*d^2*c^5-32*e^2*d*c^3+32*e*d
^3*c^2+(-16*d^5+24*e^4)*c-48*e^3*d^2))*a+(8*e^2*d*b^5+(-12*e^2*c^2-24*e*d^2*c-2*
d^4)*b^4+(32*e*d*c^3+16*d^3*c^2-32*e^3*c+16*e^2*d^2)*b^3+(-8*e*c^5-20*d^2*c^4+80
*e^2*d*c^2-32*e*d^3*c+(-8*d^5-20*e^4))*b^2+(8*d*c^6-24*e^2*c^4-32*e*d^2*c^3+24*d
^4*c^2+32*e^3*d*c+16*e^2*d^3)*b+(-c^8+16*e*d*c^5-8*d^3*c^4-16*e^3*c^3-8*e^2*d^2*
c^2+8*e*d^4*c+(-4*d^6-8*e^4*d))))*x^2+(4*e^6*a^2+(-16*e^5*d*b+(-8*e^5*c^2+24*e^4
*d^2*c-8*e^3*d^4))*a+((-8*e^5*c+12*e^4*d^2)*b^2+(24*e^4*d*c^2-32*e^3*d^3*c+(8*e^
2*d^5-8*e^6))*b+(2*e^4*c^4-16*e^3*d^2*c^3+20*e^2*d^4*c^2+(-8*e*d^6+16*e^5*d)*c+(
d^8-8*e^4*d^3))))*x-e^8

####################################################
det(x*I-M^9)
= x^5+(-a^9-9*b*a^7-9*c*a^6+(-27*b^2-9*d)*a^5+(-45*c*b-9*e)*a^4+(-30*b^3-36*d*b-18
*c^2)*a^3+(-54*c*b^2-27*e*b-27*d*c)*a^2+(-9*b^4-27*d*b^2-27*c^2*b+(-18*e*c-9*d^2
))*a+(-9*c*b^3-9*e*b^2-18*d*c*b+(-3*c^3-9*e*d)))*x^4+((-9*e^2*b+(-18*e*d*c-3*d^3
))*a^6+(27*e*d*b^2+(27*e*c^2+27*d^2*c)*b+(9*d*c^3-27*e^2*c-27*e*d^2))*a^5+((-36*
e*c-18*d^2)*b^3+(-54*d*c^2-27*e^2)*b^2+(-9*c^4+54*e*d*c+9*d^3)*b+(36*e*c^3+54*d^
2*c^2-54*e^2*d))*a^4+(9*e*b^5+45*d*c*b^4+(30*c^3+63*e*d)*b^3+(-27*e*c^2-27*d^2*c
)*b^2+(-99*d*c^3-54*e^2*c+27*e*d^2)*b+(-9*c^5+108*e*d*c^2+63*d^3*c-30*e^3))*a^3+
(-9*d*b^6-27*c^2*b^5+(-54*e*c-27*d^2)*b^4+(54*d*c^2-27*e^2)*b^3+(54*c^4+81*e*d*c
-27*d^3)*b^2+(27*e*c^3-81*d^2*c^2-27*e^2*d)*b+(-54*d*c^4+27*e^2*c^2+135*e*d^2*c+
18*d^4))*a^2+(9*c*b^7+18*e*b^6+27*d*c*b^5+(-45*c^3+54*e*d)*b^4+(-135*e*c^2+27*d^
2*c)*b^3+(27*d*c^3-108*e^2*c+54*e*d^2)*b^2+(27*c^5+27*e*d*c^2-18*d^3*c-36*e^3)*b
+(9*e*c^4-63*d^2*c^3+54*e^2*d*c+45*e*d^3))*a+(-b^9-9*d*b^7+9*c^2*b^6+(27*e*c-27*
```

```
d^2)*b^5+(27*d*c^2+18*e^2)*b^4+(-18*c^4+63*e*d*c-30*d^3)*b^3+(-63*e*c^3+27*d^2*c
^2+27*e^2*d)*b^2+(9*d*c^4-54*e^2*c^2+54*e*d^2*c-9*d^4)*b+(3*c^6-9*e*d*c^3-18*d^3
*c^2-9*e^3*c+27*e^2*d^2)))*x^3+((-9*e^4*c-18*e^3*d^2)*a^4+(-18*e^4*b^2+(18*e^3*d
*c+63*e^2*d^3)*b+(30*e^3*c^3-27*e^2*d^2*c^2-9*e*d^4*c+(-3*d^6-45*e^4*d)))*a^3+(6
3*e^3*d*b^3+(-27*e^3*c^2-81*e^2*d^2*c-54*e*d^4)*b^2+(27*e^2*d*c^3+27*e*d^3*c^2+(
27*d^5-54*e^4)*c+135*e^3*d^2)*b+(-27*e^2*c^5+27*e*d^2*c^4-18*d^4*c^3+54*e^3*d*c^
2-27*e^2*d^3*c+(-27*e*d^5-27*e^5)))*a^2+((-9*e^3*c-54*e^2*d^2)*b^4+(27*e^2*d*c^2
+99*e*d^3*c+(9*d^5+9*e^4))*b^3+(27*e^2*c^4-54*e*d^2*c^3-54*d^4*c^2+27*e^3*d*c-10
8*e^2*d^3)*b^2+(-27*e*d*c^5+45*d^3*c^4+63*e^3*c^3-81*e^2*d^2*c^2+54*e*d^4*c+(18*
d^6+54*e^4*d))*b+(9*e*c^7-9*d^2*c^6-54*e^2*d*c^4+63*e*d^3*c^3+(-27*d^5+27*e^4)*c
^2+27*e^3*d^2*c-54*e^2*d^4))*a+(-3*e^3*b^6+(27*e^2*d*c+9*e*d^3)*b^5+(-18*e^2*c^3
-54*e*d^2*c^2-9*d^4*c+9*e^3*d)*b^4+(45*e*d*c^4+30*d^3*c^3-63*e^3*c^2-27*e^2*d^2*
c+36*e*d^4)*b^3+(-9*e*c^6-27*d^2*c^5+135*e^2*d*c^3-27*e*d^3*c^2+(-27*d^5-54*e^4)
*c-27*e^3*d^2)*b^2+(9*d*c^7-27*e^2*c^5-54*e*d^2*c^4+36*d^4*c^3+108*e^3*d*c^2-54*
e^2*d^3*c+(27*e*d^5-9*e^5))*b+(-c^9+18*e*d*c^6-9*d^3*c^5-18*e^3*c^4-27*e^2*d^2*c
^3+27*e*d^4*c^2+(-9*d^6+36*e^4*d)*c-30*e^3*d^3)))*x^2+(-9*e^6*d*a^2+((-18*e^6*c+
27*e^5*d^2)*b+(27*e^5*d*c^2-36*e^4*d^3*c+(9*e^3*d^5-9*e^7)))*a+(-3*e^6*b^3+(27*e
^5*d*c-18*e^4*d^3)*b^2+(9*e^5*c^3-54*e^4*d^2*c^2+45*e^3*d^4*c+(-9*e^2*d^6+18*e^6
*d))*b+(-9*e^4*d*c^4+30*e^3*d^3*c^3+(-27*e^2*d^5+9*e^6)*c^2+(9*e*d^7-27*e^5*d^2)
*c+(-d^9+9*e^4*d^4))))*x-e^9.
```

Next, knowing the fact that the recurrences for each term are of order $\binom{5}{4}$, $\binom{5}{3}$, $\binom{5}{2}$, and $\binom{5}{1}$, respectively, we compute these using a computer algebra system.

```
#################################################
The recurrence for $-1$ is $e^n$.

#################################################
The recurrence of v_n where
v_n are the coefficients of -x^1
of the det(X*I-M^n):

v(n) = -d*v(n-1)
 -(c*e)*v(n-2)
 -(b*e^2)*v(n-3)
 +(-e^3*a)*v(n-4)
 +e^4*v(n-5).

###############################################
The recurrence of z_n where
z_n are the coefficients of -x^2
of the det(X*I-M^n):

z(n) = c*z(n-1)
 +(a*e-b*d)*z(n-2)
 +(e*(b^2+d)-a*(2*c*e-d^2))*z(n-3)
 +(e^2*(a^2+b)+d^3-d*e*(a*b+3*c))*z(n-4)
 +(e*(2*e^2+2*a*d*e+e*c*(a^2+2*b)-b*d^2))*z(n-5)
 +(e^2*(d^2-c*e-a^3*e-3*a*b*e+a*c*d))*z(n-6)
```

```
 +(-e^3*(d*a^2+a*e-c^2+2*b*d))*z(n-7)
 +(-e^4*(a*c+d))*z(n-8)
 +(-e^5*b)*z(n-9)
 +(-e^6)*z(n-10).


###############################################
The recurrence of y_n where
y_n are the coefficients of -x^3
of the det(X*I-M^n):

y(n) = -b*y(n-1)
 +(-a*c-d)*y(n-2)
 +(-a*e+c^2-a^2*d-2*b*d)*y(n-3)
 +(-a^3*e-3*a*b*e+a*c*d-c*e+d^2 )*y(n-4)
 +(2*e^2+e*(2*a*d+a^2*c+2*b*c)-b*d^2 )*y(n-5)
 +(e^2*(a^2+b)+e*(-3*c*d-a*b*d)+d^3)*y(n-6)
 +(e*(b^2*e+a*d^2+d*e-2*a*c*e))*y(n-7)
 +(e^2*(a*e-b*d) )*y(n-8)
 +(c*e^3)*y(n-9)
 +(-e^4)*y(n-10).


 ###############################################
The recurrence of x_n where
x_n are the coefficients of x^4
of the det(X*I-M^n):

x(n) = a*x(n-1)
 +(b)*x(n-2)
 +(c)*x(n-3)
 +(d)*x(n-4)
 +(e)*x(n-5).
```

Acknowledgement

The authors would like to thank Paul Young for pointing out reference [6] and the fact that the recurrences $R_k(i,r)$ are sequences of order $\binom{r}{i}$. This helped greatly in the proofs.

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MSC2010: 11A55

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