Statistics of Domino Tilings on a Rectangular Board

Thotsaporn 'Aek' Thanatipanonda

Science Division, Mahidol University International College,
Nakornphathom, Thailand
thotsaporn@gmail.com

July 3, 2018

Introduction

It is well known the Fibonacci sequence, F_n, is the number of ways to cover a 2-by-$(n-1)$ board using only the horizontal(H) or vertical(V) 2-by-1 dominos.

It is natural to generalize this idea to a rectangular m-by-n board where m is a fixed number and n is symbolic.

We can try harder and consider the mixed moment $E[V^a H^b]$ for fixed non-negative integers a, b but general m, n. After all these moments will give an information of the distribution of "V-H statistics".

Fibonacci numbers and Statistics on an 2-by-n board

We fixed the board to size 2-by-n.
Let V_n be a random variable of the number of vertical dominos on the tiling of the board of size 2-by-n.

First consider the power sum, $S[V^r] := \sum_{b \in B_n} V(b)^r$.
For $r = 0$, the number of possible tilings on the 2-by-n board, $S[V^0] = f_n$, where f_n is the $(n+1)$-th Fibonacci numbers.

Next consider the straight moment,
$E[V^r] = S[V^r]/S[V^0] = S[V^r]/f_n$.

The average number of V, vertical domino, on a 2-by-n board,
$\mu_n := E[V] = S[V]/f_n$.

Fibonacci numbers and Statistics on an 2-by-n board

Eventually we are interested in the moment about the mean, $E[(V-\mu)^r]$ and then the scaled-moments $\dfrac{E[(V-\mu)^r]}{E[(V-\mu)^2]^{r/2}}$ (which can show the normality of an asymptotic distribution).

Fast Calculation

The generating function of a random variable V is defined by

$$F_n(v) := \sum_{b \in B_n} v^{V(b)}.$$

For example, on the board of size 2-by-3, $F_3(v) = v^3 + 2v.$

$S[V^0] = F_n(1)$ and $S[V^r]$ is obtained by applying the operator $(v\frac{d}{dv})^r$ to $F_n(v)$, then substitute $v = 1$.

Fast Calculation

Next, we define the grand generating function $H(v,t)$ by

$$H(v,t) = \sum_{n=0}^{\infty} F_n(v) t^n.$$

In this problem, we have

$$H(v,t) = \frac{1}{1 - vt - t^2}, \quad (1)$$

which can be derived from a simple recurrence

$$H(v,t) = 1 + vtH(v,t) + t^2 H(v,t).$$

Fast Calculation

With equation (1) in hand, we calculate data of $S[V^r]$ (and hence $E[V^r]$) very fast. Recall that

$$\sum_{n=0}^{\infty} S[V^0]t^n = H(1,t) = \frac{1}{1-t-t^2}.$$

In general,

$$\sum_{n=0}^{\infty} S[V^r]t^n = (v\frac{d}{dv})^r H(v,t)|_{v=1}.$$

Fast Calculation

From the quotient rule in calculus, we find the fact that

$$\sum_{n=0}^{\infty} S[V^r]t^n = (v\frac{d}{dv})^r H(v,t)|_{v=1} = \frac{P_r(t)}{(1-t-t^2)^{r+1}}.$$

where $P_r(t)$ is a polynomial in t of degree at most $2r$.

Hence $S[V^r]$ in fact satisfies the recurrence of the form $(N^2 - N - 1)^{r+1}$ and can be written in the form,

$$S[V^r] = A(n)f_n + B(n)f_{n-1}$$

where $A(n)$ and $B(n)$ are polynomial in n of degree at most r.

Conjectures

We then be able to use computer program to conjecture these formulas by trying to fit the polynomial to the data. Given $\phi = \dfrac{1+\sqrt{5}}{2}$.

$$f_n := S[V^0] = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

$$\mu_n := E[V] = \frac{1}{f_n}\left[\frac{n}{5}f_n + \frac{2}{5}(n+1)f_{n-1}\right] \approx \frac{n}{5} + \frac{2}{5\phi}(n+1),$$

$$E[V^2] = \frac{1}{f_n}\left[\frac{n(5n+12)}{25}f_n + \frac{4(n+1)}{25}f_{n-1}\right]$$
$$\approx \frac{n(5n+12)}{25} + \frac{4(n+1)}{25\phi}.$$

Conjectures

From these calculations, we are able to make a general conjecture of $E[V^r]$ starting from the leading terms (and after simplify ϕ):

Conjectures

$$E[V^r]$$
$$\approx \frac{n^r}{5^{r/2}}\Bigg[1 + \frac{r(2r+\sqrt{5}-3)}{\sqrt{5}n} + \frac{r(r-1)(6r^2-32r+6\sqrt{5}r+37-9\sqrt{5})}{15n^2}$$
$$+\frac{2\sqrt{5}r(r-1)(r-2)(2r^3+3\sqrt{5}r^2-23r^2-16\sqrt{5}r+77r+16\sqrt{5}-73)}{75n^3}+$$
$$\frac{r(r-1)(r-2)(r-3)(6r^4-120r^3+12\sqrt{5}r^3-138\sqrt{5}r^2+800r^2-2100r+}{225n^4}$$
$$+\dots\Bigg],$$

Conjectures

We use the conjectures of straight moments to calculate the moment about the mean:

$$E[(V-\mu)^0] = 1,$$

$$E[(V-\mu)^1] = 0,$$

$$E[(V-\mu)^2] = E[X^2] - E[X]^2 = \frac{4\sqrt{5}n + 4\sqrt{5} - 8}{25},$$

$$E[(V-\mu)^3] = E[X^3] - 3E[X^2]E[X] + 2E[X]^3 = \frac{8\sqrt{5}n}{125} + \frac{8\sqrt{5}}{125} - \frac{48}{125},$$

$$E[(V-\mu)^4] = \frac{48n^2}{125} + \frac{96n}{125} - \frac{272\sqrt{5}n}{625} + \frac{16}{25} - \frac{272\sqrt{5}}{625},$$

$$E[(V-\mu)^5] \approx \frac{64n^2}{125} + \frac{128n}{125} - \frac{736\sqrt{5}n}{625} + \frac{9776}{3125} - \frac{144\sqrt{5}}{125},$$

$$E[(V-\mu)^6] \approx \frac{192\sqrt{5}n^3}{625} + \dots,$$

Conjectures

These data lead us to the conjectures of general formulas of moment about the mean which we will prove formally.

$$E[(V-\mu)^{2r}] = \left(\frac{2}{5\sqrt{5}}\right)^r \frac{(2r)!n^r}{r!} + \text{ smaller terms,}$$

$$E[(V-\mu)^{2r+1}] = \frac{2}{15}\left(\frac{2}{5\sqrt{5}}\right)^r \frac{(2r+1)!n^r}{(r-1)!} + \text{ smaller terms.}$$

Toward the Proof

We will prove the conjectures on $E[V^r]$ and $E[(V-\mu)^r]$ from the previous slide. Then use them to conclude the asymptotic normality distribution of V. The method of guess and check will again play an important role here.

The generating function $F_n(v)$ can be defined by the recurrence

$$F_n(v) = vF_{n-1}(v) + F_{n-2}(v) \quad , n \geq 2,$$

where

$$F_0(v) = 1, \quad F_1(v) = v.$$

Toward the Proof

The centralized probability generating function of $F_n(v)$ is

$$G_n(v) := \sum_i p(i) v^{i-\mu} = \frac{1}{f_n v^{\mu}} F_n(v),$$

The recurrence will look like

$$G_n(v) = v \frac{f_{n-1} G_{n-1}(v)}{f_n v^{\mu_n - \mu_{n-1}}} + \frac{f_{n-2} G_{n-2}(v)}{f_n v^{\mu_n - \mu_{n-2}}} \quad , n \geq 2, \tag{2}$$

where

$$G_0(v) = 1, \quad G_1(v) = 1.$$

We will use this recurrence to set up some relations for the proof.

Probability Background Interim

But before diving into the proof, we need to discuss some necessary probability background.

Definition (Exponential moment generating function)

$$\phi(t) := E[e^{tX}] = \begin{cases} \sum_x e^{tx} p(x) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & \text{if } x \text{ is continuous.} \end{cases}$$

Note $m_n := E[X^n] = \phi^n(0)$.

Probability Background Interim

We then have that

$$\phi(t) := E[e^{tX}] = \sum_x \sum_n \frac{t^n x^n}{n!} p(x) = \sum_n \frac{t^n}{n!} \sum_x x^n p(x) = \sum_n \frac{t^n}{n!} m_n.$$

Moment of the standard normal distribution:

$$\phi(t) = e^{\frac{t^2}{2}} = \sum_r \frac{t^{2r}}{r!2^r} \text{ implies that } m_{2r} = \frac{(2r)!}{r!2^r} \text{ and } m_{2r-1} = 0.$$

Main Theorem

Back to the main theorem

Theorem

Let $E_r(n) = \dfrac{E[(V-\mu)^r]}{r!}$ *then*

$$E_{2r}(n) = \left(\frac{2}{5\sqrt{5}}\right)^r \frac{n^r}{r!} + \text{ smaller terms},$$

$$E_{2r+1}(n) = \frac{2}{15}\left(\frac{2}{5\sqrt{5}}\right)^r \frac{n^r}{(r-1)!} + \text{ smaller terms}$$

The Proof

Proof We see that

$$G_n(e^t) = \sum_i p(i)e^{t(i-\mu)} = \phi(t), \quad \text{where the random variable } X = i - \mu.$$

Now define the Maclaurin series of $G_n(e^t)$ by

$$G_n(e^t) = \sum_r E_r(n)t^r.$$

By the fact of probability generating function mentioned earlier, $E_r(n) = \dfrac{E[(V-\mu)^r]}{r!}$.

The Proof

The recurrence (2) becomes:

$$G_n(e^t) = e^t \frac{f_{n-1} G_{n-1}(e^t)}{f_n e^{t(\mu_n - \mu_{n-1})}} + \frac{f_{n-2} G_{n-2}(e^t)}{f_n e^{t(\mu_n - \mu_{n-2})}} \quad , n \geq 2. \tag{3}$$

We apply an induction on r using (3) to show the assertion of $E_r(n)$.

$$G_n(e^t) = \frac{f_{n-1}}{f_n} \left[\sum_{r=0}^{\infty} \frac{(1 - \mu_n + \mu_{n-1})^r t^r}{r!} \right] \cdot \left[\sum_{r=0}^{\infty} E_r(n-1) t^r \right] + \frac{f_{n-2}}{f_n} \left[\sum_{r=0}^{\infty} \frac{(-\mu_n + \mu_{n-2})^r t^r}{r!} \right] \cdot \left[\sum_{r=0}^{\infty} E_r(n-2) t^r \right].$$

The Proof

By comparing coefficient of t^r, we obtain the relations:

$$
\begin{aligned}
& E_r(n) - \frac{f_{n-1}}{f_n} E_r(n-1) - \frac{f_{n-2}}{f_n} E_r(n-2) \\
& = \frac{f_{n-1}}{f_n} a_n E_{r-1}(n-1) + \frac{f_{n-2}}{f_n} b_n E_{r-1}(n-2) \\
& + \frac{f_{n-1}}{f_n} \frac{a_n^2}{2} E_{r-2}(n-1) + \frac{f_{n-2}}{f_n} \frac{b_n^2}{2} E_{r-2}(n-2) \\
& + \frac{f_{n-1}}{f_n} \frac{a_n^3}{6} E_{r-3}(n-1) + \frac{f_{n-2}}{f_n} \frac{b_n^3}{6} E_{r-3}(n-2) + \dots .
\end{aligned}
$$

where $a_n = 1 - \mu_n + \mu_{n-1} \approx 1 - \frac{1}{\sqrt{5}}$ and $b_n = -\mu_n + \mu_{n-2} \approx -\frac{2}{\sqrt{5}}$.

The Proof

Case1: even case:

Left hand side:

$$E_{2r}(n) - \frac{f_{n-1}}{f_n}E_{2r}(n-1) - \frac{f_{n-2}}{f_n}E_{2r}(n-2)$$
$$= \left(\frac{f_{n-1}}{f_n} + 2\frac{f_{n-2}}{f_n}\right)\left(\frac{2}{5\sqrt{5}}\right)^r \frac{n^{r-1}}{(r-1)!} + \text{ smaller terms}$$

The Proof

Right hand side is

$$\frac{f_{n-1}}{f_n} a_n E_{2r-1}(n-1) + \frac{f_{n-2}}{f_n} b_n E_{2r-1}(n-2) +$$
$$\frac{f_{n-1}}{f_n} \frac{a_n^2}{2} E_{2r-2}(n-1) + \frac{f_{n-2}}{f_n} \frac{b_n^2}{2} E_{2r-2}(n-2) + \text{ smaller terms}$$
$$= (\frac{f_{n-1}}{f_n} \frac{a_n^2}{2} + \frac{f_{n-2}}{f_n} \frac{b_n^2}{2}) \left(\frac{2}{5\sqrt{5}}\right)^{r-1} \frac{n^{r-1}}{(r-1)!} + \text{ smaller terms.}$$

which are equal.

The Proof

Case2: odd case: . . . (can be done similarly).

Asymptotic Distribution

To show the normality of V, we show that $\frac{V_n - \mu_n}{\sigma_{V_n}} \sim N(0,1)$ as $n \to \infty$.

We verify, as $n \to \infty$, $m_{2r} = \frac{(2r)!}{2^r r!}$ for every r and $m_{2r-1} = 0$.

Asymptotic Distribution

Corollary

The distribution of number of V on a tiling of 2-by-n board is asymptotically normal.

Proof.

$$\frac{E[(V_n-\mu_n)^{2r}]}{E[(V_n-\mu_n)^2]^r} = \frac{\left(\frac{2}{5\sqrt{5}}\right)^r \frac{n^r}{r!}(2r)!}{\left(\frac{4n}{5\sqrt{5}}\right)^r} = \frac{(2r)!}{2^r r!},$$

and

$$\frac{E[(V_n-\mu_n)^{2r+1}]}{E[(V_n-\mu_n)^2]^{(r+1/2)}} = 0,$$

□

Asymptotic Distribution

Remark The conjectures of the straight moment $E[V^r]$ can be verified similarly with less calculation.

Statistics on an m-by-n board

The method we used for a 2-by-n board can be generalized to an m-by-n board for a fixed m and symbolic n.

References

Doron Zeilberger, *Automated Derivation of Limiting Distributions Of Combinatorial Random Variables Whose Generating Functions are Rational*, The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger, Dec. 24, 2016.

Doron Zeilberger, *Automatic Count Tilings*, The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger, Jan. 20, 2006.