A Quantitative Study on Average Number of Spins of Two-Player Dreidel

Thotsaporn "Aek" Thanatipanonda

Mahidol University, International College

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Mathematics and Culture

Mathematics although a pure science has come with culture.
Styles and methods for research mathematics change over the time.

Motivation: Gambler's Ruin

- A game starts with a gambler who has zero chips.
- For each play, s/he could gain one chip or lose one chip with equal probability $p = q = 1/2$.
- The game continues until the gambler gains M chips or loses N chips.
- We are interested in the expected number of plays until the game ends.

Solution to Gambler's Ruin

Let $G(a)$ be the expected number of plays until the game ends when the gambler initially has a chips. The system of recurrence relations that G satisfies is as follows:

$$G(M) = G(-N) = 0,$$

$$G(a) = 1 + \frac{1}{2}G(a+1) + \frac{1}{2}G(a-1), -N < a < M.$$

Solution to Gambler's Ruin

The solution to this system is well known, i.e. [2],

$$G(a) = (N + a) \cdot (M - a), \quad -N \leq a \leq M. \tag{1}$$

This beautiful result is a cornerstone of probability models and is the source of inspiration to our research problem.

Rule for Dreidel

A dreidel is a four-sided spinning top, and the game associated with it is played during the Jewish holiday of Hanukkah. Each side of the dreidel bears a letter of the Hebrew alphabet: gimel, hay, nun and shin.

Rule for Dreidel

- Gimel: Player takes the whole pot, after which everyone donates one nut to the pot and then the next person spins.
- Hay: Player takes the smaller half of the pot and the next person spins.
- Nun: Player takes and gives nothing and the next person spins.
- Shin: Player gives one nut to the pot and the next person spins.

The game ends when only one person has all of the nuts in their possession.

Analysis of Simplified Dreidel

We consider the simplest non-trivial version where two players spin with only two outcomes: Gimel (player takes the whole pot) and Shin (player gives one to the pot), and we answer the question of how long the game will last if the two players start with a and b nuts, respectively.

Analysis of Simplified Dreidel

Let $D(a, p, b)$ denote the average number of spins until the game ends where the first player has a nuts, the second player has b nuts, and p nuts are in the pot. We can derive the following recurrences for the simplified game:

$$D(a, p, b) = 1 + \frac{1}{2}\left[D(b-1, 2, a+p-1) + D(b, p+1, a-1)\right], \qquad (2)$$
$$D(0, p, b) = D(a, p, 0) = 0, \qquad \text{for any } a \geq 0,\ p \geq 0,\ b \geq 0.$$

Analysis of Simplified Dreidel

We are interested in the average number of spins until the game ends,

$$T(a, b) := D(a, 2, b).$$

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Approximation

$$T(a, b) \approx \widetilde{T}(a, b) = \frac{12}{19}ab + c_2 a + (c_2 + \frac{2}{19})b + c_0,$$

in which

$$c_2 = -0.30463656275164039697189322263 5\dots$$

and

$$c_0 = 2.1310261721834108187045214415 6\dots.$$

This approximation is astonishing, for example, $|T(100, 100) - \widetilde{T}(100, 100)| \leq 10^{-12}$.

Nice Observation but Rigorous?

We will rewrite the system of equations (2) recursively in terms of $T(a, b)$ only.

$$T(a,b) = \sum_{i=0}^{min(2a-2,2b-1)} \left(\frac{1}{2}\right)^i + \sum_{i=1}^{min(a,b)} \frac{T(b-i, a+i)}{2^{2i-1}} \qquad \text{(Key)}$$
$$+ \sum_{i=2}^{min(a,b+1)} \frac{T(a-i, b+i)}{2^{2i-2}}.$$

Dreidel conjecture

We notice that each of the sequences

$$\square_a := Dr(a+1, p, b) - Dr(a, p, b)$$

and

$$\square_b := Dr(a, p, b+1) - Dr(a, p, b).$$

converge to constants as before.

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Denote $Q(a, b) := Dr(a, 2, b)$.

Conjecture (Dreidel conjecture)

$$\begin{aligned} Q(a,b) \approx \widetilde{Q}(a,b) &= 2.2181415186261818190483262884\,3 \cdot ab \\ &- 1.0970966703340591066947863966\,9 \cdot a \\ &- 0.4470795446435881356886522681\,82 \cdot b \\ &+ 2.8388078373423186967598713586\,8. \end{aligned}$$

Application

On the practical perspective, we may assume that it takes 10 seconds per play, if both players start with 10 nuts, an average game will last 28.10 minutes. And if both players start with 15 nuts, an average game will last 69.33 minutes.

References

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