The Leading Quadratic Constant in Two-Player Dreidel

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Abstract

For the full four-outcome two-player Dreidel recurrence underlying Thanatipanonda's 2019 conjecture, we prove the leading quadratic asymptotic for the expected number of spins. If the player to move has a tokens, the pot initially contains a fixed number $p_0 \geq 1$ of tokens, and the other player has b tokens, let σ be the first time either player's stack becomes zero. We show

$$\mathbb{E}_{a,p_0,b}\sigma = \frac{4}{v}ab + O\big((a+b)\log(a+b)\big),$$

where v is the effective variance of a corrected martingale driven by an autonomous, geometrically mixing pot chain. The constant v admits a regenerative representation and can be enclosed rigorously by rational arithmetic. We obtain

$$2.21814151862747687227681625 < \frac{4}{v} < 2.21814151862747687227681832.$$

Thus the leading ab coefficient in Thanatipanonda's 2019 full-game conjecture is

$$2.2181415186274768722768 2\ldots.$$

The proof uses an anti-Poisson corrector for the alternating wealth imbalance, a second Poisson corrector for its quadratic variation, and a logarithmic bound on the pot size at the random exit time.

1 Introduction

The expected duration of Dreidel has been studied both probabilistically and experimentally. Robinson and Vijay proved that, for a fixed number of players, the expected duration is $O(n^2)$ when each player begins with n tokens [1]. For two players, Thanatipanonda later obtained high-precision formulas for a simplified game and proposed a numerical bilinear approximation for the full four-outcome game [2]. In the notation of that paper, with

$$Q(a,b) = Dr(a,2,b),$$

the fitted leading coefficient was

$$2.218141518626181818904832628843.$$

The full-game formula was stated as a conjecture.

The purpose of this paper is to identify and prove the leading quadratic coefficient. The main observation is that, before either player reaches zero tokens, the pot coordinate evolves autonomously and mixes rapidly because every Gimel resets the pot to 2. The difference between the two players'

stacks is not itself a martingale: each spin swaps the roles of the players and introduces a pot-dependent drift. An anti-Poisson corrector removes this drift exactly. The resulting martingale has an effective variance v, and a second Poisson corrector turns its quadratic variation into an exact stopped identity. This identity is the analogue of the classical gambler's-ruin formula, with v replacing the unit variance.

The proof gives more than an order bound. For each fixed initial pot p_0,

$$\mathbb{E}\sigma = \frac{N^2 - (a-b)^2}{v} + O(N \log N), \qquad N = a + b + p_0,$$

and hence

$$\mathbb{E}\sigma = \frac{4}{v}ab + O(N \log N).$$

The coefficient $4/v$ is therefore intrinsic: it is the inverse effective diffusion variance of the corrected wealth imbalance.

The use of Poisson equations and martingale correctors for additive functionals of Markov chains is standard in stochastic-process theory; see, for example, Meyn–Tweedie [4] and Glynn–Meyn [5]. The argument here is elementary because the pot chain has an explicit reset regeneration.

Stopping convention

There are two conventions in the literature. Robinson and Vijay use the traditional rule that a player leaves only when required to contribute a token and unable to do so [1]. Thanatipanonda's prose describes the endpoint differently, but the supplementary Maple routine used to generate the full-game conjecture explicitly imposes the boundary condition

$$Dr(0, p, b) = Dr(a, p, 0) = 0,$$

and its transition routine is exactly the four-outcome recurrence used below [3]. Our main theorem therefore matches the recurrence actually used to generate the 2019 conjecture. The traditional payment-failure model is related to it by an exact one-token shift; see Section 11.

2 The two-player chain

A state is written

$$(a, p, b),$$

where the player to move owns a tokens, the pot contains $p \geq 1$ tokens, and the other player owns b tokens. While $a, b \geq 1$, the four equally likely transitions are

$$G : (a, p, b) \mapsto (b - 1, 2, a + p - 1), \tag{1}$$
$$H : (a, p, b) \mapsto (b, \lceil p/2 \rceil, a + \lfloor p/2 \rfloor), \tag{2}$$
$$N : (a, p, b) \mapsto (b, p, a), \tag{3}$$
$$S : (a, p, b) \mapsto (b, p + 1, a - 1). \tag{4}$$

The total

$$N = a + p + b$$

is conserved by every interior transition. Define

$$\sigma = \inf\{t : a_t = 0 \text{ or } b_t = 0\}.$$

We will prove the following.

Theorem 2.1 (Leading asymptotic)**.** *Fix $p_0 \geq 1$. Uniformly for $a, b \geq 1$ as $a + b \to \infty$,*

$$\mathbb{E}_{a,p_0,b}\sigma = \frac{4}{v}ab + O\big((a+b)\log(a+b)\big),$$

where $v \in (0, \infty)$ is defined in (9)*. In particular, if $a, b \to \infty$ with both comparable to $a + b$, then*

$$\frac{\mathbb{E}_{a,p_0,b}\sigma}{ab} \longrightarrow \frac{4}{v}.$$

3 The autonomous pot chain

Let

$$X = a - b.$$

From (1)–(4), one spin transforms X as

$$X' = -X + c_J(p),$$

where

$$c_G(p) = -p, \qquad c_H(p) = -\lfloor p/2 \rfloor, \qquad c_N(p) = 0, \qquad c_S(p) = 1.$$

The pot evolves independently of X:

$$p \mapsto 2, \quad \lceil p/2 \rceil, \quad p, \quad p+1$$

with probability 1/4 each. Let

$$(Kf)(p) = \frac{1}{4}\left[f(2) + f(\lceil p/2 \rceil) + f(p) + f(p+1)\right].$$

Write

$$K = \frac{1}{4}R + \frac{3}{4}Q,$$

where R sends every state to 2, while Q is the conditional non-Gimel kernel

$$p \mapsto \lceil p/2 \rceil, \ p, \ p+1$$

with probabilities 1/3 each.

Proposition 3.1 (Regenerative stationary law)**.** *The pot chain has a unique stationary distribution π, given by*

$$\pi = \sum_{t=0}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^t \delta_2 Q^t. \tag{5}$$

Moreover, for $m \geq 2$,

$$\pi\{p \geq m\} \leq \left(\frac{3}{4}\right)^{m-2}.$$

Proof. The series in (5) is a probability measure. Using $K = \frac{1}{4}R + \frac{3}{4}Q$ and the fact that $\nu R = \delta_2$ for every probability measure ν, a direct calculation gives $\pi K = \pi$. Uniqueness follows by coupling two chains with the same sequence of faces: at the first Gimel both jump to 2 and thereafter agree, so

$$\|K^t(p,\cdot) - K^t(q,\cdot)\|_{\mathrm{TV}} \leq (3/4)^t.$$

Under Q, starting from 2, the pot can increase by at most one per step. Hence $\delta_2 Q^t\{p \geq m\} = 0$ for $t < m - 2$, and summing the remaining weights in (5) proves the tail bound. □

In particular, π has moments of every polynomial order.

4 An anti-Poisson corrector

The one-step conditional drift term is

$$\mu(p) = \mathbb{E}[c_J(p) \mid p] = \frac{1 - p - \lfloor p/2 \rfloor}{4}.$$

We seek r satisfying

$$r + Kr = -\mu. \tag{6}$$

Lemma 4.1 (Linear corrector)**.** *Equation* (6) *has a solution of the form*

$$r(p) = \frac{3}{13}p + s(p),$$

where s is bounded. In fact $\|s\|_\infty \le 15/13$.

Proof. Substitution gives

$$(I + K)s = d,$$

where

$$d(p) = \begin{cases} -11/26, & p \text{ even}, \\ -15/26, & p \text{ odd}. \end{cases}$$

Equivalently,

$$s(p) = \frac{4d(p) - s(2) - s(\lceil p/2 \rceil) - s(p+1)}{5}.$$

The right-hand side defines a contraction on ℓ^∞ with factor $3/5$. Banach's fixed-point theorem gives a unique bounded solution. Starting the iteration at zero gives

$$\|s\|_\infty \le \frac{\|\mathcal{T}0\|_\infty}{1 - 3/5} = \frac{15}{13}.$$

□

Define, while the process remains interior,

$$Y_t = (-1)^t \big(X_t + r(P_t)\big). \tag{7}$$

Lemma 4.2 (Corrected martingale)**.** $(Y_t)_{t<\sigma}$ *is a martingale. If*

$$w(p) = \frac{1}{4} \sum_{J \in \{G,H,N,S\}} \left(c_J(p) + r(g_J(p)) + r(p)\right)^2, \tag{8}$$

where $g_J(p)$ is the next pot under face J, then

$$Y_t^2 - \sum_{j<t} w(P_j)$$

is also a martingale up to σ.

Proof. Using $X_{t+1} = -X_t + c_J(P_t)$,

$$Y_{t+1} - Y_t = (-1)^{t+1} \left(c_J(P_t) + r(P_{t+1}) + r(P_t)\right).$$

Its conditional mean vanishes by (6). Equation (8) is therefore its conditional second moment, proving the second assertion. □

Lemma 4.1 also identifies the growth of w.

Lemma 4.3 (Variance growth)**.** *As $p \to \infty$,*

$$w(p) = \frac{44}{169}p^2 + O(p).$$

Proof. The four corrected increments have leading coefficients

$$-\frac{10}{13}p, \qquad -\frac{2}{13}p, \qquad \frac{6}{13}p, \qquad \frac{6}{13}p,$$

respectively. Averaging their squares gives $44p^2/169 + O(p)$. □

Define the effective variance

$$v = \pi(w) = \sum_{p \geq 1} \pi(p) w(p). \tag{9}$$

Proposition 3.1 and Lemma 4.3 show that $v < \infty$. Also $v > 0$: at $p = 2$, the Gimel and Nun branches end at the same pot but their c-values differ by 2, so their corrected increments cannot both vanish; moreover $\pi(2) > 0$.

5 A second corrector

Set

$$g(p) = w(p) - v.$$

Let $J_1, J_2, \ldots$ be the successive Dreidel outcomes driving the autonomous pot chain, and set

$$R_G = \min\{m \geq 1 : J_m = G\}.$$

Thus $P_0 = p$, the transition from P_{m-1} to P_m is driven by J_m, and R_G is the number of pre-reset states visited, including P_0. Define

$$h(p) = \mathbb{E}_p \left[\sum_{t=0}^{R_G - 1} g(P_t) \right]. \tag{10}$$

Lemma 5.1 (Second Poisson equation)**.** *The function h satisfies*

$$h(p) = O(1 + p^2)$$

and

$$h - Kh = w - v. \tag{11}$$

Proof. Before the first Gimel, $P_t \leq p + t$. Since $g(p) = O(1 + p^2)$ and $\mathbb{P}(R_G > t) = (3/4)^t$, the series (10) is absolutely integrable and $h(p) = O(1 + p^2)$.

For the normalization at the reset state,

$$h(2) = \sum_{t=0}^{\infty} \left(\frac{3}{4}\right)^t (\delta_2 Q^t) g = 4\pi(g) = 0,$$

using (5) and $\pi(g) = 0$. A first-step decomposition of (10) then gives

$$h(p) = g(p) + \frac{1}{4}h(\lceil p/2 \rceil) + \frac{1}{4}h(p) + \frac{1}{4}h(p+1),$$

because the Gimel branch terminates the sum. Since $h(2) = 0$, this is exactly (11). □

6 The stopped martingale identity

For each fixed total N, the interior state space is finite. From every interior state there is a positive-probability path to the boundary: two consecutive Shin outcomes, whenever the first has not already hit the boundary, transform

$$(a, p, b) \mapsto (a - 1, p + 2, b - 1).$$

Repeated pairs therefore reach a zero stack. Since the interior state space at fixed N is finite and no nonempty subset of interior states is closed, all interior states are transient for the killed chain. Consequently σ is almost surely finite and $\mathbb{E}\sigma < \infty$.

Proposition 6.1 (Exact stopped identity)**.** *For every interior starting state,*

$$v\mathbb{E}\sigma = \mathbb{E}[Y_\sigma^2] - Y_0^2 - h(P_0) + \mathbb{E}[h(P_\sigma)]. \tag{12}$$

Moreover, uniformly over states of total N,

$$\mathbb{E}\sigma = O(N^2).$$

Proof. Put $\tau_n = \sigma \wedge n$. For fixed N, $P_{t\wedge\sigma}$, $X_{t\wedge\sigma}$, $r(P_{t\wedge\sigma})$, $h(P_{t\wedge\sigma})$, and $w(P_{t\wedge\sigma})$ are uniformly bounded. Since τ_n is bounded, optional stopping applied to the martingale in Lemma 4.2 gives

$$\mathbb{E}[Y_{\tau_n}^2] - Y_0^2 = \mathbb{E}\sum_{t<\tau_n} w(P_t).$$

As $n \to \infty$, the left side converges by bounded convergence and the right side by monotone convergence, yielding

$$\mathbb{E}[Y_\sigma^2] - Y_0^2 = \mathbb{E}\sum_{t<\sigma} w(P_t).$$

From (11), the stopped process

$$M_t^{(h)} = h(P_{t\wedge\sigma}) + \sum_{j=0}^{t\wedge\sigma-1} (w(P_j) - v)$$

is a martingale. Therefore, for every n,

$$\mathbb{E}h(P_{\tau_n}) - h(P_0) + \mathbb{E}\sum_{j<\tau_n} (w(P_j) - v) = 0.$$

Letting $n \to \infty$, bounded convergence applies to the h-term, while dominated convergence applies to the sum because $\mathbb{E}\sigma < \infty$ and w is bounded on the fixed-N state space. Hence

$$\mathbb{E}\sum_{t<\sigma} w(P_t) = v\mathbb{E}\sigma + h(P_0) - \mathbb{E}h(P_\sigma).$$

Combining these identities gives (12). Since $P_\sigma, |X_\sigma| \le N$, Lemmas 4.1 and 5.1 make every term on the right $O(N^2)$, and $v > 0$. □

7 The pot at the random exit time

The final ingredient is that the pot is typically logarithmic even though the total mass is N.

Lemma 7.1 (Exit-pot moments)**.** *Let the initial pot be p_0 and $N = a + p_0 + b$. Then*

$$\mathbb{E}P_\sigma \le p_0 + C\log N + C$$

and

$$\mathbb{E}P_\sigma^2 \le C\big(p_0^2 + \log^2 N + 1\big),$$

for an absolute constant C.

Proof. Couple the game to an unrestricted pot chain $\widehat{P}_t$ driven by the same infinite sequence of Dreidel outcomes and evolving by the kernel K even after the game stops. Up to time σ we have $\widehat{P}_t = P_t$, in particular $\widehat{P}_\sigma = P_\sigma$. Set $B = \max(p_0, 2)$. For deterministic time t, let A_t be the number of consecutive non-Gimel transitions immediately preceding t in this infinite outcome sequence, with $A_t = t$ if no Gimel has yet occurred. Pathwise,

$$\widehat{P}_t \le B + A_t.$$

For arbitrary L, k,

$$\mathbb{P}(P_\sigma > B + k) \le \mathbb{P}(\sigma > L) + \mathbb{P}(\exists t \le L : A_t \ge k).$$

Proposition 6.1 and Markov's inequality give

$$\mathbb{P}(\sigma > L) \le CN^2/L,$$

while a union bound gives

$$\mathbb{P}(\exists t \le L : A_t \ge k) \le L(3/4)^k.$$

Take $L = N^4$ and

$$k_0 = \left\lceil \frac{8\log N}{\log(4/3)} \right\rceil.$$

Then the probability that $P_\sigma > B + k_0$ is $O(N^{-2})$. Since $P_\sigma \le N$, splitting according to this event gives the two moment bounds. □

8 Proof of the main theorem

At time σ, one stack is zero, so

$$|X_\sigma| = N - P_\sigma.$$

Using Lemma 4.1,

$$Y_\sigma^2 = N^2 + O\left(N(P_\sigma + 1) + P_\sigma^2 + 1\right).$$

Lemma 7.1 therefore gives

$$\mathbb{E}Y_\sigma^2 = N^2 + O\left(N(p_0 + \log N + 1) + p_0^2 + \log^2 N\right). \tag{13}$$

Also,

$$Y_0^2 = (a-b)^2 + O\left(N(p_0 + 1) + p_0^2 + 1\right),$$

and Lemmas 5.1 and 7.1 give

$$\mathbb{E}|h(P_\sigma)| = O(p_0^2 + \log^2 N + 1).$$

Substitution in (12) yields

$$\mathbb{E}\sigma = \frac{N^2 - (a-b)^2}{v} + O\left(N(p_0 + \log N + 1) + p_0^2 + \log^2 N\right). \tag{14}$$

For fixed p_0 this is

$$\mathbb{E}\sigma = \frac{N^2 - (a-b)^2}{v} + O(N \log N).$$

Finally,

$$N^2 - (a-b)^2 = 4ab + 2p_0(a+b) + p_0^2,$$

which proves Theorem 2.1.

9 Certified evaluation of the constant

The representation (5) gives

$$v = \sum_{t=0}^{\infty} \frac{1}{4} \left(\frac{3}{4}\right)^t (\delta_2 Q^t) w. \tag{15}$$

This can be enclosed without truncating the infinite pot chain.

Let s_n be the nth fixed-point iterate from the proof of Lemma 4.1, starting at 0. Since the contraction factor is $3/5$,

$$\|s - s_n\|_\infty \le \frac{15}{13} \left(\frac{3}{5}\right)^n. \tag{16}$$

Thus every $r(p)$ and $w(p)$ appearing in a finite prefix of (15) can be enclosed by exact rational arithmetic.

For the tail, $|s(p)| \le 15/13$ and $g_J(p) \le p + 1$ imply

$$|c_J(p) + r(g_J(p)) + r(p)| \le p + \frac{3(p+1) + 15 + 3p + 15}{13} = \frac{19p + 33}{13}.$$

Thus the slightly looser convenient bound

$$w(p) \le \left(\frac{19p + 49}{13}\right)^2$$

holds for every $p \ge 1$. Under Q, starting from 2, after t steps one has $P_t \le t + 2$. Therefore the omitted contribution after time L is bounded above by

$$\sum_{t=L+1}^{\infty} \frac{1}{4} \left(\frac{3}{4}\right)^t \left(\frac{19(t+2) + 49}{13}\right)^2,$$

an explicitly summable geometric-polynomial tail.

Using $L = 230$ and 150 corrector iterations gives the rigorous interval below. The accompanying script uses exact rational arithmetic for the enclosure and directed decimal rounding when printing its endpoints.

$$1.80331145078384659274980 5 < v < 1.80331145078384659274980 7,$$

and hence

$$2.218141518627476872276816 < \frac{4}{v} < 2.218141518627476872276819.$$

In particular,

$$\boxed{\frac{4}{v} = 2.2181415186274768722768\ldots}$$

to the displayed rounded digits.

As an independent check, exact rational finite-state computations using the same zero-stack recurrence give the mixed differences

$$\Delta_a\Delta_b Q(n,n)$$

shown below:

n	$\Delta_a\Delta_b Q(n,n)$
10	$2.218141477982962893\ldots$
15	$2.218141518612687561\ldots$
20	$2.218141518627472506\ldots$

which is consistent with convergence to the certified value above.

10 Relation to the 2019 conjecture

Thanatipanonda's supplementary program defines the full-game solver 'DrSolve' by declaring any state with first or third coordinate zero to have value zero and otherwise averaging the four transitions (1)–(4) plus one spin [3]. The least-squares fit for the conjecture is then performed on $15 \le a, b \le 25$ [3, 2], and the stored fitted coefficient is

$$2.218141518626181818904832628843.$$

Theorem 2.1 proves that the asymptotic coefficient of ab is instead

$$\frac{4}{v} = 2.2181415186274768722768\ldots.$$

The difference, approximately 1.295×10^{-12}, is small enough to be compatible with the remarkable finite-range accuracy reported in [2]. Our result proves the leading quadratic term only. It does not establish the conjectured linear coefficients, constant term, or exponentially small remainder.

11 Exact relation to the traditional payment-failure rule

Let $D_0(a,p,b)$ denote the expectation under the zero-stack boundary condition used above, and let $D_{\mathrm{pf}}(a,p,b)$ denote the expectation under the traditional rule in which a player with zero tokens remains in the game until a required payment cannot be made. For the two-player chain the two models are exactly related.

Proposition 11.1 (One-token shift)**.** *For all $a, b \ge 0$ and $p \ge 1$,*

$$D_{\mathrm{pf}}(a,p,b) = D_0(a+1,p,b+1).$$

Proof. Set $F(a,p,b) = D_0(a+1,p,b+1)$. Compare the four one-step equations for F with those of the payment-failure game. For Hay and Nun the shifted successor is exactly the successor of the payment-failure state with one token added to each player. For Shin, if $a > 0$ the same statement holds, while if $a = 0$ the payment-failure game absorbs and the shifted zero-stack chain lands on a state whose third coordinate is zero. The Gimel case is analogous: if $b > 0$ the successors agree after shifting, while if $b = 0$ both models absorb on that outcome. Thus F satisfies exactly the Bellman equations of D_{pf}. On each fixed-total class the absorbing linear system has a unique solution, proving the identity. □

Consequently the traditional model has the same leading quadratic coefficient without any boundary-delay estimate:

$$D_{\mathrm{pf}}(a,p_0,b) = \frac{4}{v}(a+1)(b+1) + O((a+b)\log(a+b)) = \frac{4}{v}ab + O((a+b)\log(a+b))$$

for fixed p_0. Proposition 11.1 also explains a useful computational check: mixed differences computed with the payment-failure convention are the zero-stack mixed differences shifted by one in each player coordinate.

12 Discussion

The coefficient in the two-player Dreidel asymptotic is naturally interpreted as an effective diffusion constant. The pot is a fast, geometrically mixing environment; the alternating player imbalance is a Markov-additive process in that environment. The anti-Poisson corrector converts it into a martingale, while the second corrector replaces the random accumulated conditional variance by its stationary average v. The stopped identity then produces the quadratic gambler's-ruin scale exactly.

Several questions remain. First, the much stronger numerical bilinear approximation in [2] suggests that the $O(N \log N)$ remainder is far from sharp. Second, exact mixed differences appear to converge extremely rapidly, hinting at a boundary-layer or spectral explanation. Finally, analogous corrector methods may be useful for multi-player Dreidel, though the slow variables are then multidimensional.

Related-work status

Robinson–Vijay prove the quadratic order bound, while Thanatipanonda states the full four-outcome approximation as a conjecture. As of September 2026, we are not aware of a published proof of the leading coefficient obtained here. Searches of arXiv/web-indexed literature, current citation metadata for the 2019 article, and the author's publication list did not reveal a later resolution. This statement is intentionally limited to the literature located by those searches rather than a claim of exhaustive priority.

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Computational experiments were used to discover the corrector and effective-variance structure and to independently check the resulting constant. OpenAI ChatGPT was used substantially during exploratory calculations, code generation, literature-search assistance, and drafting and editing of the manuscript. The author reviewed the resulting arguments and computations and takes full responsibility for the mathematical claims and final text. The proof is stated in exact probabilistic identities and explicit analytic bounds.

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