THE CURIOUS BOUNDS OF FLOOR FUNCTION SUMS

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Abstract. The sums of floor functions have been studied by Jacobsthal, Carlitz, Grimson and more recently by Tverberg. In 2017, Onphaeng and Pongsriiam proved some sharp upper and lower bounds for the sums of Jacobsthal and Tverberg. In this paper, we formulate a concise formula for the sums and then use it to give proofs of the upper and lower bounds that were claimed by Tverberg. Furthermore, we conjecture all of the bounds related to this kind of problem.

1. Introduction

In 1957, Jacobsthal [3] defined and studied a function of the form

$$f_m(\{a_1,a_2\},k) = \left\lfloor \frac{a_1+a_2+k}{m} \right\rfloor - \left\lfloor \frac{a_1+k}{m} \right\rfloor - \left\lfloor \frac{a_2+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor$$

for fixed $m \in \mathbb{Z}^+$ with $a_1, a_2, k \in \mathbb{Z}$. Then, he defined the partial sums

$$S_m(\{a_1,a_2\},K) = \sum_{k=0}^{K} f_m(\{a_1,a_2\},k), \qquad 0 \leq a_1, a_2, K \leq m-1.$$

Jacobsthal, then later Carlitz [1], Grimson [2] and Tverberg [5] all proved that $S_m(\{a_1,a_2\},K) \geq 0$. In 2012, Tverberg [5] proposed a generalized notation for the partial sums for any set $A = \{a_1, ..., a_n\}$ with $0 \leq a_1, ..., a_n, K \leq m-1$ and $n = |A|$, that is,

$$S_m(\{a_1,...,a_n\},K) = \sum_{k=0}^{K} \sum_{T \subset [1,n]} (-1)^{n-|T|} \left\lfloor \frac{k + \sum_{i \in T} a_i}{m} \right\rfloor .$$

He also claimed without proof the other upper and lower bounds of S_m for sets $\{a_1,a_2\}$ and $\{a_1,a_2,a_3\}$ (i.e. $n = 2,3$). We would like to investigate the bounds for S_m for all $n \in \mathbb{Z}^+$. In 2017, Onphaeng and Pongsriiam [4] furnished a proof for the upper bounds when n is even and ≥ 4 and the lower bounds when n is odd and ≥ 3. In this paper, we supply the missing proofs of Tverberg's upper bounds. Furthermore, we conjecture all the bounds of S_m not previously mentioned . We summarize the findings for these sums in the chart below.

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n	Lower Bound	Lower Bound Credit	Upper Bound	Upper Bound Credit
1	0	Trivial	$m-1$	Trivial
2	0	Jacobsthal; Carlitz; Grimson; Tverberg **Section 2**	$\lfloor \frac{m}{2} \rfloor$	Tverberg **Section 2**
3	$-2\lfloor \frac{m}{2} \rfloor$	Tverberg; Onphaeng, Pongsriiam	$\lfloor \frac{m}{3} \rfloor$	Tverberg **Section 3**
4	$-3\lfloor \frac{m}{3} \rfloor$ (Conjecture)	**Section 4**	$4\lfloor \frac{m}{2} \rfloor$	Onphaeng, Pongsriiam
odd (≥ 5)	$-2^{n-2}\lfloor \frac{m}{2} \rfloor$	Onphaeng, Pongsriiam	(Conjectures)	**Section 5**
even (≥ 5)	(Conjectures)	**Section 5**	$2^{n-2}\lfloor \frac{m}{2} \rfloor$	Onphaeng, Pongsriiam

We discuss the case of $n = 1$ here as it sets the foundation for the main strategy that we use in this paper to prove higher cases. We begin with an explicit definition of Jacobsthal's sum.

Definition 1.1. For any $m \in \mathbb{Z}$ and $a_1, K \in \mathbb{Z}^+ \cup \{0\}$,

$$S_m(\{a_1\}, K) = \sum_{k=0}^{K} \left(\left\lfloor \frac{a_1 + k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \right).$$

The sum can be written concisely without using the summation symbol.

Proposition 1.2. *For* $0 \leq K \leq m-1$ *and* $a_1 \in \mathbb{Z}^+ \cup \{0\}$,

$$S_m(\{a_1\}, K) = \left\lfloor \frac{a_1}{m} \right\rfloor (K+1) + \max(0, a_1 \bmod m + K - m + 1).$$

Proof. We observe

$$\left\lfloor \frac{a_1 + k}{m} \right\rfloor = \left\lfloor \frac{a_1}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor + \left\lfloor \left\{ \frac{a_1}{m} \right\} + \left\{ \frac{k}{m} \right\} \right\rfloor,$$

where $\{x\} = x - \lfloor x \rfloor$. Furthermore, since $0 \leq k \leq m-1$,

$$\left\lfloor \left\{ \frac{a_1}{m} \right\} + \left\{ \frac{k}{m} \right\} \right\rfloor = \left\lfloor \frac{a_1 \bmod m}{m} + \frac{k}{m} \right\rfloor.$$

The result is derived as follows:

$$\sum_{k=0}^{K}\left(\left\lfloor\frac{a_1+k}{m}\right\rfloor-\left\lfloor\frac{k}{m}\right\rfloor\right)$$

$$=\sum_{k=0}^{K}\left(\left\lfloor\frac{a_1}{m}\right\rfloor+\left\lfloor\frac{k}{m}\right\rfloor+\left\lfloor\left\{\frac{a_1}{m}\right\}+\left\{\frac{k}{m}\right\}\right\rfloor-\left\lfloor\frac{k}{m}\right\rfloor\right)$$

$$=\left\lfloor\frac{a_1}{m}\right\rfloor(K+1)+\sum_{k=0}^{K}\left\lfloor\frac{a_1\text{mod } m}{m}+\frac{k}{m}\right\rfloor$$

$$=\left\lfloor\frac{a_1}{m}\right\rfloor(K+1)+\sum_{k=0}^{m-(a_1\text{mod } m)-1}\left\lfloor\frac{a_1\text{mod } m+k}{m}\right\rfloor$$

$$+\sum_{k=m-a_1\text{mod } m}^{K}\left\lfloor\frac{a_1\text{mod } m+k}{m}\right\rfloor$$

$$=\left\lfloor\frac{a_1}{m}\right\rfloor(K+1)+\sum_{k=0}^{m-(a_1\text{mod } m)-1}0+\sum_{k=m-a_1\text{mod } m}^{K}1$$

$$=\left\lfloor\frac{a_1}{m}\right\rfloor(K+1)+\max(0,(a_1 \bmod m)+K-m+1)$$

□

The bounds of $S_m(\{a_1\},K)$ are easily attained from Proposition 1.2.

Corollary 1.3. *For* $0\leq a_1,K\leq m-1$,

$$0\leq S_m(\{a_1\},K)\leq m-1.$$

In particular, the maximum occurs precisely when $a_1=K=m-1$.

Proof. The result follows from the fact that

$$S_m(\{a_1\},K)=\max(0,a_1+K-m+1).$$

□

In the following sections, we use similar derivations to show bounds for $S_m(\{a_1,...a_n\},K)$ when $n>1$.

2. Lower and Upper Bounds for $n=2$

The lower bound for $n=2$ has been shown by multiple people (see [1], [2], [3], and [5]), while the upper bound was mentioned by Tverberg in [5]. Like in the case of $n=1$, we introduce a new form for the sum by generalizing Proposition 1.2 and then prove its upper bound. We also use the form to provide a new proof for its lower bound. Like before, we write out the sum of Jacobsthal explicitly.

Definition 2.1. For any $m \in \mathbb{Z}^+$ and any $a_1, a_2, K \in \mathbb{Z}^+ \cup \{0\}$,

$$S_m(\{a_1, a_2\}, K) = \sum_{k=0}^{K} \left(\left\lfloor \frac{a_1 + a_2 + k}{m} \right\rfloor - \left\lfloor \frac{a_1 + k}{m} \right\rfloor - \left\lfloor \frac{a_2 + k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor \right).$$

We show that the sum can be written concisely without using the summation symbol, similar to Proposition 1.2.

Proposition 2.2. *For $0 \leq K \leq m-1$, and any $a_1, a_2 \in \mathbb{Z}^+ \cup \{0\}$,*

$$\begin{aligned} S_m(\{a_1, a_2\}, K) = &\left(\left\lfloor \frac{a_1 + a_2}{m} \right\rfloor - \left\lfloor \frac{a_1}{m} \right\rfloor - \left\lfloor \frac{a_2}{m} \right\rfloor \right)(K+1) \\ &+ \max(0, (a_1 + a_2) \bmod m + K - m + 1) \\ &- \max(0, a_1 \bmod m + K - m + 1) \\ &- \max(0, a_2 \bmod m + K - m + 1). \end{aligned}$$

Proof. Rewriting the two-variable sum in Definition 2.1 as a series of one-variable sums,

$$S_m(\{a_1, a_2\}, K) = S_m(\{a_1 + a_2\}, K) - S_m(\{a_1\}, K) - S_m(\{a_2\}, K),$$

allows us to apply Proposition 1.2 to each sum to get our result. □

A symmetry exists in Proposition 2.2. We outline the pattern in the lemma below. This, along with a partial result in Theorem 2.4, gives us the desired upper and lower bounds in Corollary 2.5 and Theorem 2.6, respectively.

Lemma 2.3. *(Mirrored Sums) For $0 \leq a_1, a_2 \leq m-1$ and $0 \leq K \leq m-2$,*

$$S_m(\{a_1, a_2\}, K) = S_m(\{m - a_1, m - a_2\}, m - 2 - K).$$

Proof. It is enough to show the claim for $0 \leq a_1 + a_2 \leq m$. Otherwise, we have that $(m - a_1) + (m - a_2) < m$, in which case we can use a similar argument by substituting a_1 with $m - a_1$ and a_2 with $m - a_2$. For the case $a_1 = a_2 = 0$, the result trivially holds by Definition 2.1. For the case $0 < a_1 + a_2 < m$, Proposition 2.2 simplifies to

$$\begin{aligned} &S_m(\{a_1, a_2\}, K) \\ &= \max(0, a_1 + a_2 + K - m + 1) \\ &\quad - \max(0, a_1 + K - m + 1) - \max(0, a_2 + K - m + 1). \end{aligned} \tag{2.1}$$

Furthermore, we note that $m < 2m - (a_1 + a_2) < 2m$, which gives

$$\begin{aligned} &S_m(\{m - a_1, m - a_2\}, m - 2 - K) \\ &= m - 1 - K + \max(0, m - (a_1 + a_2) - K - 1) \\ &\quad - \max(0, m - a_1 - K - 1) - \max(0, m - a_2 - K - 1). \end{aligned} \tag{2.2}$$

Now consider the following equations that use the fact that

$$\max(0,x)-\max(0,-x)=x, \text{ for all } x\in\mathbb{R}.$$

$$\begin{aligned}&\max(0,a_1+a_2+K-m+1)-\max(0,-a_1-a_2-K+m-1)\\&\quad=a_1+a_2+K-m+1\end{aligned}\tag{2.3}$$

$$\begin{aligned}&\max(0,m-a_1-K-1)-\max(0,a_1+K-m+1)\\&\quad=m-a_1-K-1\end{aligned}\tag{2.4}$$

$$\begin{aligned}&\max(0,m-a_1-K-1)-\max(0,a_2+K-m+1)\\&\quad=m-a_2-K-1.\end{aligned}\tag{2.5}$$

Then, (2.3)+(2.4)+(2.5) confirms (2.1) = (2.2).

Finally, we consider $a_1+a_2=m$. Here, Proposition 2.2 gives

$$\begin{aligned}&S_m(\{a_1,a_2\},K)\\&\quad=K+1-\max(0,a_1+K-m+1)\\&\qquad-\max(0,a_2+K-m+1)\end{aligned}\tag{2.6}$$

$$\begin{aligned}&S_m(\{m-a_1,m-a_2\},m-2-K)\\&\quad=m-(K+1)-\max(0,m-a_1-K-1)\\&\qquad-\max(0,m-a_2-K-1).\end{aligned}\tag{2.7}$$

In this case, (2.4)+(2.5) confirms (2.6) = (2.7). This concludes the proof. □

We show the upper bound for half the range of K using differences.

Theorem 2.4. *For $0\le a_1,a_2\le m-1$ and $0\le K\le\left\lfloor\frac{m}{2}\right\rfloor-1$,*

$$S_m(\{a_1,a_2\},K)\le\left\lfloor\frac{m}{2}\right\rfloor.$$

Proof. We show the stronger result, that for $0\le K\le\left\lfloor\frac{m}{2}\right\rfloor-1$,

$$S_m(\{a_1,a_2\},K)\le K+1.$$

$\underline{K=0}$:

$$\begin{aligned}S_m(\{a_1,a_2\},0)=&\left\lfloor\frac{a_1+a_2}{m}\right\rfloor+\max(0,(a_1+a_2)\bmod m-m+1)\\&-\max(0,a_1-m+1)-\max(0,a_2-m+1)\\=&\left\lfloor\frac{a_1+a_2}{m}\right\rfloor+0-0-0\\\le&1\end{aligned}$$

$\underline{1\le K\le\left\lfloor\frac{m}{2}\right\rfloor-1}$: It is enough to show that

$$\Delta_m:=S_m(\{a_1,a_2\},K)-S_m(\{a_1+1,a_2\},K-1)\le 1.$$

By using Proposition 2.2 we explicitly write out the two sums:

$$\begin{aligned}
S_m(\{a_1,a_2\},K) = &\left\lfloor \frac{a_1+a_2}{m} \right\rfloor (K+1) \\
&+\max(0,(a_1+a_2) \bmod m + K - m + 1) \\
&-\max(0,a_1+K-m+1) \\
&-\max(0,a_2+K-m+1), \\
S_m(\{a_1+1,a_2\},K-1) = &\left\lfloor \frac{a_1+a_2+1}{m} \right\rfloor K \\
&+\max(0,(a_1+a_2+1) \bmod m + K - m) \\
&-\max(0,a_1+K-m+1) \\
&-\max(0,a_2+K-m).
\end{aligned}$$

We determine Δ_m in four cases according to the possible values of a_1+a_2 and $a_2+K-m+1$.

Case 1: **(**$a_1+a_2 < m$ **and** $a_2+K-m+1 \le 0$**)** If $a_1+a_2 < m-1$, then both sums are the same:

$$\begin{aligned}
S_m(\{a_1,a_2\},K) =&0+\max(0,a_1+a_2+K-m+1) \\
&-\max(0,a_1+K-m+1)-0, \\
S_m(\{a_1+1,a_2\},K-1) =&0+\max(0,(a_1+1)+a_2+K-m) \\
&-\max(0,(a_1+1)+K-m)-0.
\end{aligned}$$

If $a_1+a_2 = m-1$, then both sums evaluate to K. Therefore, we get $\Delta_m = 0$ for this case.

Case 2: **(**$a_1+a_2 < m$ **and** $a_2+K-m+1 > 0$**)** Again, we assume $a_1+a_2 < m-1$:

$$\begin{aligned}
S_m(\{a_1,a_2\},K) =&0+\max(0,a_1+a_2+K-m+1) \\
&-\max(0,a_1+K-m+1)-(a_2+K-m+1), \\
S_m(\{a_1+1,a_2\},K-1) =&0+\max(0,(a_1+1)+a_2+K-m+1) \\
&-\max(0,(a_1+1)+K-m)-(a_2+K-m).
\end{aligned}$$

Therefore, $\Delta_m = S_m(\{a_1,a_2\},K) - S_m(\{a_1+1,a_2\},K-1) = -1$. A similar argument holds for $a_1+a_2 = m-1$.

Case 3: **(**$a_1+a_2 \ge m$ **and** $a_2+K-m+1 \le 0$**)**

$$\begin{aligned}
S_m(\{a_1,a_2\},K) =&(K+1)+\max(0,(a_1+a_2-m)+K-m+1) \\
&-\max(0,a_1+K-m+1)-0
\end{aligned}$$

$$\begin{aligned}S_m(\{a_1+1,a_2\},K-1) =& K+\max(0,(a_1+1+a_2-m)+K-m)\\ &-\max(0,(a_1+1)+K-m)-0\end{aligned}$$

Therefore, $\Delta_m = S_m(\{a_1,a_2\},K) - S_m(\{a_1+1,a_2\},K-1) = +1$.

Case 4: ($a_1 + a_2 \geq m$ and $a_2 + K - m + 1 > 0$) $\Delta_m = 0$ using similar reasoning to Case 2 and Case 3.

We summarize the four cases for $0 < K \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$ in the table below.

Case	$a_1 + a_2$	$a_2 + K - m + 1$	Δ_m
1	$< m$	≤ 0	0
2	$< m$	> 0	-1
3	$\geq m$	≤ 0	$+1$
4	$\geq m$	> 0	0

TABLE 1. Summary of Δ_m values.

This shows that $-1 \leq \Delta_m \leq 1$. Thus, we have shown that

$$S_m(\{a_1,a_2\},K) \leq S_m(\{a_1,a_2\},K-1)+1 \leq K+1 \leq \left\lfloor \frac{m}{2} \right\rfloor$$

for all $0 \leq K \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$. ∎

We apply Lemma 2.3 and Theorem 2.4 to show the upper bound of S_m.

Corollary 2.5. *For* $0 \leq a_1, a_2, K \leq m-1$,

$$S_m(\{a_1,a_2\},K) \leq \left\lfloor \frac{m}{2} \right\rfloor.$$

Proof. For $0 \leq K \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$, Theorem 2.4 gives the result. For $\left\lfloor \frac{m}{2} \right\rfloor \leq K \leq m-2$, $S_m(\{a_1,a_2\},K) = S_m(\{m-a_1, m-a_2\}, m-2-K)$ by Lemma 2.3. From there, we apply Theorem 2.4 to the right hand side and get the result. Finally, for $K = m-1$, it is easily seen that $S_m(\{a_1,a_2\},K) = 0$. This completes the proof. ∎

The lower bound is now easy to show using Δ_m.

Theorem 2.6. *For* $0 \leq a_1, a_2, K \leq m-1$,

$$0 \leq S_m(\{a_1,a_2\},K).$$

Proof. Without loss of generality, we assume that $0 \leq a_2 \leq a_1 \leq m-1$. Consider $0 \leq K \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$. Thus, the conditions of Case 2 (i.e. $a_1 + a_2 < m$ and $a_2 + K - m + 1 > 0$) cannot be met because if $a_1 + a_2 < m$, then $a_2 \leq \left\lfloor \frac{m}{2} \right\rfloor$. So, $a_2 + K - m + 1 \leq 0$. This shows that $\Delta_m \neq -1$, which means $\Delta_m = 0$ or 1. This, along with $S_m(\{a_1,a_2\},0) \geq 0$, gives us the lower bound as desired. Next, we consider $\lfloor \frac{m}{2} \rfloor \le K \le m-2$, and apply Lemma 2.3 to complete the argument. Lastly, $S_m(\{a_1, a_2\}, m-1) = 0$. We now have the complete result. □

3. Upper Bound for $n = 3$

In this section, we follow the previous style of rewriting the sum, observing its symmetry and using a difference to prove the upper bound. The lower bound has already been proven in [4]. This time, we use Tverberg's formulation to write out the sum explicitly.

Definition 3.1. For any $m \in \mathbb{Z}^+$ and $a_1, a_2, a_3, K \in \mathbb{Z}^+ \cup \{0\}$,

$$\begin{aligned} S_m(\{a_1,a_2,a_3\},K) = \sum_{k=0}^{K} \Bigg(& \left\lfloor \frac{a_1+a_2+a_3+k}{m} \right\rfloor \\ & - \left\lfloor \frac{a_1+a_2}{m} \right\rfloor - \left\lfloor \frac{a_2+a_3}{m} \right\rfloor - \left\lfloor \frac{a_1+a_3}{m} \right\rfloor \\ & + \left\lfloor \frac{a_1}{m} \right\rfloor + \left\lfloor \frac{a_2}{m} \right\rfloor + \left\lfloor \frac{a_3}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \Bigg). \end{aligned}$$

Like before, we show that the sum can be written concisely without using the summation symbol.

Proposition 3.2. *For $0 \le K \le m-1$ and $a_1, a_2, a_3 \in \mathbb{Z}^+ \cup \{0\}$,*

$$\begin{aligned} S_m(\{a_1,a_2,a_3\},K) = & \Bigg(\left\lfloor \frac{a_1+a_2+a_3}{m} \right\rfloor - \left\lfloor \frac{a_1+a_2}{m} \right\rfloor - \left\lfloor \frac{a_2+a_3}{m} \right\rfloor \\ & - \left\lfloor \frac{a_1+a_3}{m} \right\rfloor + \left\lfloor \frac{a_1}{m} \right\rfloor + \left\lfloor \frac{a_2}{m} \right\rfloor + \left\lfloor \frac{a_3}{m} \right\rfloor \Bigg)(K+1) \\ & + \max(0, (a_1+a_2+a_3) \bmod m + K - m + 1) \\ & - \max(0, (a_1+a_2) \bmod m + K - m + 1) \\ & - \max(0, (a_2+a_3) \bmod m + K - m + 1) \\ & - \max(0, (a_1+a_3) \bmod m + K - m + 1) \\ & + \max(0, a_1 \bmod m + K - m + 1) \\ & + \max(0, a_2 \bmod m + K - m + 1) \\ & + \max(0, a_3 \bmod m + K - m + 1). \end{aligned}$$

Proof. We can rewrite our three-variable sum in Definition 3.1 as two-variable sums:

$$\begin{aligned} S_m(\{a_1,a_2,a_3\},K) = & S_m(\{a_1,a_2+a_3\},K) \\ & - S_m(\{a_1,a_2\},K) - S_m(\{a_1,a_3\},K) \end{aligned} \tag{3.1}$$

We then apply Proposition 2.2 to each sum to get our result. □

A symmetry exists for $S_m(\{a_1, a_2, a_3\}, K)$, similar to Lemma 2.3.

Lemma 3.3. *(Mirrored Sums) For* $0 \le a_1, a_2, a_3 \le m-1$ *and* $0 \le K \le m-2$,

$$S_m(\{a_1, a_2, a_3\}, K) = S_m(\{m-a_1, m-a_2, m-a_3\}, m-2-K).$$

Proof. The three-variable sum can be rewritten as a series of two-variable sums and we can reason as follows:

$$\begin{aligned}
&S_m(\{a_1, a_2, a_3\}, K)\\
&=S_m(\{a_1, a_2+a_3\}, K) - S_m(\{a_1, a_2\}, K) - S_m(\{a_1, a_3\}, K)\\
&=S_m(\{a_1, (a_2+a_3) \text{mod } m\}, K) - S_m(\{a_1, a_2\}, K) - S_m(\{a_1, a_3\}, K)\\
&=S_m(\{m-a_1, m-(a_2+a_3) \text{mod } m\}, m-2-K)\\
&\quad - S_m(\{m-a_1, m-a_2\}, m-2-K)\\
&\quad - S_m(\{m-a_1, m-a_3\}, m-2-K)\\
&=S_m(\{m-a_1, (m-a_2)+(m-a_3)\}, m-2-K)\\
&\quad - S_m(\{m-a_1, m-a_2\}, m-2-K)\\
&\quad - S_m(\{m-a_1, m-a_3\}, m-2-K)\\
&=S_m(\{m-a_1, m-a_2, m-a_3\}, m-2-K).
\end{aligned}$$

The first and fifth equalities come from Equation (3.1). We take advantage of the a_i-periodicity of the sums in the second and fourth equalities. Lastly, we apply Lemma 2.3 in the third equality. □

Next, we show the upper bound for half the range of K using Δ_m.

Theorem 3.4. *For* $0 \le a_1, a_2, a_3 \le m-1$ *and* $0 \le K \le \left\lfloor \frac{m}{2} \right\rfloor - 1$,

$$S_m(\{a_1, a_2, a_3\}, K) \le \left\lfloor \frac{m}{3} \right\rfloor.$$

Proof. Without loss of generality, we assume that $0 \le a_3 \le a_2 \le a_1 \le m-1$. We break down the proof into two cases of K, that is, we would like to show

$$S_m(\{a_1, a_2, a_3\}, K) \le \begin{cases} K+1 & \text{if} \quad 0 \le K \le \left\lfloor \frac{m}{3} \right\rfloor - 1 \quad (\textbf{Case A}) \\ \left\lfloor \frac{m}{3} \right\rfloor & \text{if} \quad \left\lfloor \frac{m}{3} \right\rfloor \le K \le \left\lfloor \frac{m}{2} \right\rfloor - 1 \quad (\textbf{Case B}) \end{cases}.$$

Like previously, we will define a difference of sums:

$$\square_m := S_m(\{a_1, a_2, a_3\}, K) - S_m(\{a_1+1, a_2, a_3\}, K-1).$$

We first note that $\square_m$ can be converted to Δ_m via (3.1):

$$\begin{aligned}
\square_m =&S_m(\{a_1, a_2, a_3\}, K) - S_m(\{a_1+1, a_2, a_3\}, K-1)\\
=&[S_m(\{a_1, a_2+a_3\}, K) - S_m(\{a_1+1, a_2+a_3\}, K-1)]\\
&- [S_m(\{a_1, a_2\}, K) - S_m(\{a_1+1, a_2\}, K-1)]
\end{aligned}$$

$$-\left[S_m(\{a_1,a_3\},K)-S_m(\{a_1+1,a_3\},K-1)\right]$$
$$=\Delta_m(\{a_1,a_2+a_3\},K)-\Delta_m(\{a_1,a_2\},K)-\Delta_m(\{a_1,a_3\},K).$$

(**Case A**) From the proof of Theorem 2.6, we know that

$$\Delta_m(\{a_1,a_2\},K)\ge 0 \text{ and } \Delta_m(\{a_1,a_3\},K)\ge 0.$$

This, coupled with the fact that $\Delta_m(\{a_1,a_2+a_3\},K)\le 1$ (by Table 1) gives $\square_m \le 1$. Furthermore, with $S_m(\{a_1,a_2,a_3\},0)\le 1$, we get $S_m \le K+1$ as in the proof of Theorem 2.4.

(**Case B**) If $\square_m \le 0$, then we can use Case A (i.e. $S_m \le K+1 \le \lfloor \frac{m}{3} \rfloor$) to show

$$S_m(\{a_1,a_2,a_3\},K)\le S_m(\{a_1+1,a_2,a_3\},K-1)\le \left\lfloor \frac{m}{3}\right\rfloor.$$

If $\square_m = 1$, we show $S_m \le \lfloor \frac{m}{3} \rfloor$ directly. Observe that the only way to obtain $\square_m = 1$ is when

$$\Delta_m(\{a_1,a_2+a_3\},K)=+1,$$
$$\Delta_m(\{a_1,a_2\},K)=0,$$
$$\Delta_m(\{a_1,a_3\},K)=0.$$

This arrangement is achieved when our assumed a_1,a_2,a_3,K also satisfies all conditions from Table 1, as shown below.

Δ_m		Condition **a**	Type		Condition **b**
+1	(1a)	$a_1+(a_2+a_3) \bmod m \ge m$	AND	(1b)	$(a_2+a_3) \bmod m + K - m + 1 \le 0$
0	(2a)	$a_1+a_2 \ge m$	XOR	(2b)	$a_2+K-m+1\le 0$
0	(3a)	$a_1+a_3 \ge m$	XOR	(3b)	$a_3+K-m+1\le 0$

Keeping these conditions in mind, we bound $S_m(\{a_1,a_2,a_3\},K)$ according to whether $a_2+a_3<m$ or not.

(**Case B1**) $a_2+a_3<m$. Conditions (1a) and (1b) simplify to

$$m\le a_1+a_2+a_3<2m \text{ AND } a_2+a_3+K-m+1\le 0,$$

which therefore satisfies (2b) and (3b), implying that it also satisfies ∼(2a) and ∼(3a) by the XOR condition. Thus, we get

$$a_1+a_2<m \text{ AND } a_2+K-m+1\le 0$$
$$a_1+a_3<m \text{ AND } a_3+K-m+1\le 0.$$

Applying these conditions to Proposition 3.2 results in the following formula:

$$S_m(\{a_1,a_2,a_3\},K)$$
$$=K+1-\max(0,a_1+a_2+K-m+1)$$

$$- \max(0, a_1 + a_3 + K - m + 1) + \max(0, a_1 + K - m + 1),$$

which can be broken down into four subcases:

(B1.1): $(a_1 + K - m + 1 > 0)$

$$S_m(\{a_1, a_2, a_3\}, K) = -a_1 - a_2 - a_3 + m \le -m + m = 0$$

(B1.2): $(a_1 + K - m + 1 \le 0$ **and** $a_1 + a_3 + K - m + 1 > 0)$

$$\begin{aligned} &S_m(\{a_1, a_2, a_3\}, K) \\ &\quad = K + 1 - (a_1 + a_2 + K - m + 1) - (a_1 + a_3 + K - m + 1) \\ &\quad = (m - a_1 - a_2 - a_3) - (a_1 + K) + (m - 1) \\ &\quad \le 0 - \left\lfloor \frac{2m}{3} \right\rfloor + (m - 1) \\ &\quad \le \left\lfloor \frac{m}{3} \right\rfloor \end{aligned}$$

because $a_1 + a_2 + a_3 \ge m$ (1a) and $a_1 \ge a_2 \ge a_3$ gives $a_1 \ge \lfloor \frac{m}{3} \rfloor$. This, along with $K \ge \lfloor \frac{m}{3} \rfloor$, gives us the second to last line.

(B1.3): $(a_1 + a_3 + K - m + 1 \le 0$ **and** $a_1 + a_2 + K - m + 1 > 0)$

$$S_m(\{a_1, a_2, a_3\}, K) = m - a_1 - a_2 \le \left\lfloor \frac{m}{3} \right\rfloor$$

because $a_1 + a_2 \ge \lfloor \frac{2m}{3} \rfloor$.

(B1.4): $(a_1 + a_2 + K - m + 1 \le 0)$ This case cannot happen because $a_1 + a_2 + a_3 \ge m$ gives that $a_1 + a_2 \ge \lfloor \frac{2m}{3} \rfloor$ which means $a_1 + a_2 + K + 1 \ge m$, contradicting the condition.

We summarize these four subcases in Table 2.

Subcase	$a_1 + a_2 + K - m + 1$	$a_1 + a_3 + K - m + 1$	$a_1 + K - m + 1$	S_m
1	> 0	> 0	> 0	≤ 0
2	> 0	> 0	≤ 0	$\le \lfloor \frac{m}{3} \rfloor$
3	> 0	≤ 0	≤ 0	$\le \lfloor \frac{m}{3} \rfloor$
4	≤ 0	≤ 0	≤ 0	N/A

TABLE 2

(**Case B2**) $m \le a_2 + a_3 < 2m$. Conditions (1a) and (1b) simplify to

$$a_1 + a_2 + a_3 \ge 2m \text{ AND } a_2 + a_3 + K - 2m + 1 \le 0,$$

which therefore satisfies (2a) and (3a), implying that it also satisfies $\sim$(2b) and $\sim$(3b) by the XOR condition. Thus, we get

$$a_1 + a_2 \geq m \text{ AND } a_2 + K - m + 1 > 0$$
$$a_1 + a_3 \geq m \text{ AND } a_3 + K - m + 1 > 0.$$

Applying these conditions to Proposition 3.2 results in the following formula:

$$\begin{aligned} S_m(\{a_1, a_2, a_3\}, K) &= -(K+1) \\ &\quad - \max(0, a_1 + a_2 + K - 2m + 1) - \max(0, a_1 + a_3 + K - 2m + 1) \\ &\quad + (a_1 + K - m + 1) + (a_2 + K - m + 1) + (a_3 + K - m + 1), \end{aligned}$$

which can be broken down into three subcases:

(B2.1): ($a_1 + a_2 + K - 2m + 1 > 0$ **and** $a_1 + a_3 + K - 2m + 1 > 0$)

$$\begin{aligned} &S_m(\{a_1, a_2, a_3\}, K) \\ &= -(a_1 + a_2 + K - 2m + 1) - (a_1 + a_3 + K - 2m + 1) \\ &\quad + (a_1 + a_2 + K - 2m + 1) + (a_3 + K - m + 1) \\ &= m - a_1 \\ &\leq \left\lfloor \frac{m}{3} \right\rfloor \end{aligned}$$

because $a_1 + a_2 + a_3 \geq m$ and $a_1 \geq a_2 \geq a_3$ gives $a_1 \geq \left\lfloor \frac{2m}{3} \right\rfloor$.

(B2.2): ($a_1 + a_2 + K - 2m + 1 > 0$ **and** $a_1 + a_3 + K - 2m + 1 \leq 0$)

$$\begin{aligned} S_m(\{a_1, a_2, a_3\}, K) &= (m - a_1) + (a_1 + a_3 + K - 2m + 1) \\ &\leq m - a_1 \\ &\leq \left\lfloor \frac{m}{3} \right\rfloor \end{aligned}$$

(B2.3): ($a_1 + a_2 + K - 2m + 1 \leq 0$)

$$\begin{aligned} S_m(\{a_1, a_2, a_3\}, K) &= (m - a_1) + (a_1 + a_2 + K - 2m + 1) \\ &\quad + (a_1 + a_3 + K - 2m + 1) \\ &\leq (m - a_1) \\ &\leq \left\lfloor \frac{m}{3} \right\rfloor \end{aligned}$$

We summarize the three cases in Table 3. The results from Tables 2 and 3 completes the proof of Case B. Hence, we have proven the theorem. □

Lemma 3.3 and Theorem 3.4 give the main result, which we state as a corollary.

Subcase	$a_1+a_2+K-2m+1$	$a_1+a_3+K-2m+1$	S_m
1	>0	>0	$\leq \left\lfloor \frac{m}{3} \right\rfloor$
2	>0	≤ 0	$\leq \left\lfloor \frac{m}{3} \right\rfloor$
3	≤ 0	≤ 0	$\leq \left\lfloor \frac{m}{3} \right\rfloor$

Table 3

Corollary 3.5. *For* $0 \leq a_1, a_2, a_3, K \leq m-1$,

$$S_m(\{a_1, a_2, a_3\}, K) \leq \left\lfloor \frac{m}{3} \right\rfloor.$$

Proof. For $0 \leq K \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$, Theorem 3.4 gives us our result. For $\left\lfloor \frac{m}{2} \right\rfloor \leq K \leq m-2$, $S_m(\{a_1, a_2, a_3\}, K) = S_m(\{m-a_1, m-a_2, m-a_3\}, m-2-K)$ by Lemma 3.3. From there, we apply Theorem 3.4 to the right hand side and get the result. Finally, $S_m(\{a_1, a_2, a_3\}, m-1) = 0$. This completes the proof. □

4. (Not So Sharp) Lower Bound For $n = 4$

Given that $0 \leq a_1, a_2, a_3, a_4, K \leq m-1$, the pattern of the maximum and minimum values of $S_m(\{a_1, a_2, a_3, a_4\}, K)$ is less clear, as evidenced by some results of the computer program:

<u>Maximum Values of Sums</u>

$$\begin{aligned}[S_1, S_2, ...] =& [0, 4, 3, 8, 7, 12, 11, 16, 15, 20, 19, 24\\ & 23, 28, 27, 32, 31, 36, 35, 40, 39, 44, ...].\end{aligned}$$

<u>Minimum Values of Sums</u>

$$\begin{aligned}[S_1, S_2, ...] =& [0, 0, -3, -2, -3, -6, -5, -6, -9,\\ & -8, -9, -12, -11, -12, -15, -14,\\ & -15, -18, -17, -18, -21, -20, ...].\end{aligned}$$

It has already been shown in [4], the upper bound

$$S_m(\{a_1, a_2, a_3, a_4\}, K) \leq 4 \left\lfloor \frac{m}{2} \right\rfloor.$$

We conjecture the lower bound

$$-3 \left\lfloor \frac{m}{3} \right\rfloor \leq S_m(\{a_1, a_2, a_3, a_4\}, K).$$

In an attempt to prove this lower bound, we found that writing a difference of sums (like Δ_m or $\Box_m$) is not an efficient way to approach the problem. Accordingly, we use another method to obtain the following partial result.

Theorem 4.1. *For* $0 \leq a_1, a_2, a_3, a_4, K \leq m-1$,

$$-2\left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{m}{3} \right\rfloor \leq S_m(\{a_1,a_2,a_3,a_4\},K).$$

Proof. We can combine the following bounds from $n = 2, 3$, namely,

$$\begin{aligned} 0 &\leq S_m(\{a_1+a_2+a_3, a_4\}, K), \\ -\left\lfloor \frac{m}{2} \right\rfloor &\leq -S_m(\{a_1+a_2, a_4\}, K), \\ -\left\lfloor \frac{m}{2} \right\rfloor &\leq -S_m(\{a_1+a_3, a_4\}, K), \\ -\left\lfloor \frac{m}{3} \right\rfloor &\leq -S_m(\{a_2, a_3, a_4\}, K), \\ 0 &\leq S_m(\{a_1, a_4\}, K), \end{aligned}$$

along with the identity

$$\begin{aligned} &S_m(\{a_1,a_2,a_3,a_4\},K) \\ &= S_m(\{a_1+a_2+a_3,a_4\},K) - S_m(\{a_1+a_2,a_4\},K) \\ &\quad - S_m(\{a_1+a_3,a_4\},K) - S_m(\{a_2,a_3,a_4\},K) \\ &\quad + S_m(\{a_1,a_4\},K), \end{aligned}$$

to obtain the claimed result. □

5. Conjectures

In order to complete the analysis on this type of floor function problem, we want to show all the upper bounds and lower bounds for any number of variables, n. Onphaeng and Pongsriiam [4] were able to show the upper bound when n is even and the lower bound when n is odd.

Theorem 5.1 (Onphaeng, Pongsriiam)**.** *When n is even and m is even,*

$$S_m \leq 2^{n-2}\left\lfloor \frac{m}{2} \right\rfloor.$$

When n is odd and m is even,

$$-2^{n-2}\left\lfloor \frac{m}{2} \right\rfloor \leq S_m.$$

The bounds on both cases are obtained exactly at

$$A = \{m/2, m/2, ..., m/2\}, \quad K = m/2 - 1.$$

We conjecture the missing bounds, namely, the lower bounds when n is even and the upper bounds when n is odd. The code for these conjectures can be found on Thanatipanonda's website (www.thotsaporn.com).

Conjecture 5.2. *Suppose*

$$M(n) := \begin{cases} max\ S_m(A,K), & n\ odd \\ min\ S_m(A,K), & n\ even \end{cases}.$$

Then for $n \geq 4$,

Case 1: $n = 4k-1, 4k$ ***or*** $4k+1$, ***where*** $k \in \mathbb{Z}^+$

Under the condition that m *is a multiple of* $2k+1$, *the values of* $M(n)$ *occur exactly at*

$$A = \left\{\frac{km}{2k+1}, \frac{km}{2k+1}, ..., \frac{km}{2k+1}\right\}, K = \frac{km}{2k+1} - 1$$
$$or\ A = \left\{\frac{(k+1)m}{2k+1}, \frac{(k+1)m}{2k+1}, ..., \frac{(k+1)m}{2k+1}\right\}, K = \frac{(k+1)m}{2k+1} - 1.$$

Case 2: $n = 4k+2$, ***where*** $k \in \mathbb{Z}^+$

Under the condition that m *is a multiple of* $2k+1$ *and* $2k+3$, *the values of* $M(n)$ *occur (among other places) at*

$$A = \left\{\frac{km}{2k+1}, \frac{km}{2k+1}, ..., \frac{km}{2k+1}\right\}, K = \frac{km}{2k+1} - 1$$
$$or\ A = \left\{\frac{(k+1)m}{2k+1}, \frac{(k+1)m}{2k+1}, ..., \frac{(k+1)m}{2k+1}\right\}, K = \frac{(k+1)m}{2k+1} - 1$$
$$or\ A = \left\{\frac{(k+1)m}{2k+3}, \frac{(k+1)m}{2k+3}, ..., \frac{(k+1)m}{2k+3}\right\}, K = \frac{(k+1)m}{2k+3} - 1$$
$$or\ A = \left\{\frac{(k+2)m}{2k+3}, \frac{(k+2)m}{2k+3}, ..., \frac{(k+2)m}{2k+3}\right\}, K = \frac{(k+2)m}{2k+3} - 1.$$

Moreover, $M(n)$ *can be calculated directly from a formula similar to the equations from Propositions 2.2 and 3.2 or by*

$$M(n) = m \cdot f(n)$$

where $f(n)$ *satisfies the recurrence relation:*

$$\begin{aligned} -5(n+3)(n-2)f(n) & \\ =& 10(n^2+n-8)f(n-1) - 4(2n^2-10n+3)f(n-2) \\ & - 24(2n-11)f(n-3) - 32(2n^2-10n-1)f(n-4) \\ & - 192(n-1)(n-5)f(n-5) + 64(2n^2-22n+51)f(n-6) \\ & + 384(2n-13)f(n-7) - 256(n-3)(n-8)f(n-8) \\ & + 512(n-9)(n-8)f(n-9), \end{aligned}$$

for $n \geq 13$ *with the initial conditions*

$$f(4) = -1, f(5) = 2, f(6) = -3, f(7) = 8, f(8) = -18,$$

$$f(9) = 36, f(10) = -65, f(11) = 148, f(12) = -314.$$

For convenience, we give examples of some of the bounds (and the set A for which the values of those bounds occur) produced from the conjectures above.

- $n = 4, 5$, m is a multiple of 3:

  $n = 4$:
  $$-3 \cdot \left\lfloor \frac{m}{3} \right\rfloor \le S_m$$
  $n = 5$:
  $$S_m \le 6 \cdot \left\lfloor \frac{m}{3} \right\rfloor$$
  For these cases, $M(n)$ occurs at:
  $$A = \{m/3, m/3, ..., m/3\}, K = m/3 - 1$$
  $$\text{or } A = \{2m/3, 2m/3, ..., 2m/3\}, K = 2m/3 - 1.$$

- $n = 6$:
  - m is a multiple of 3:
    $$-9 \cdot \left\lfloor \frac{m}{3} \right\rfloor \le S_m.$$
    with the minimum at (among other places)
    $$A = \{m/3, m/3, ..., m/3\}, K = m/3 - 1$$
    $$\text{or } A = \{2m/3, 2m/3, ..., 2m/3\}, K = 2m/3 - 1.$$
  - m is a multiple of 5:
    $$-15 \cdot \left\lfloor \frac{m}{5} \right\rfloor \le S_m.$$
    with the minimum is at (among other places) at
    $$A = \{2m/5, 2m/5, ..., 2m/5\}, K = 2m/5 - 1$$
    $$\text{or } A = \{3m/5, 3m/5, ..., 3m/5\}, K = 3m/5 - 1.$$

- $n = 7, 8, 9$, m is a multiple of 5:

  $n = 7$:
  $$S_m \le 40 \cdot \left\lfloor \frac{m}{5} \right\rfloor$$
  $n = 8$:
  $$-90 \cdot \left\lfloor \frac{m}{5} \right\rfloor \le S_m$$
  $n = 9$:
  $$S_m \le 180 \cdot \left\lfloor \frac{m}{5} \right\rfloor$$

For these cases, $M(n)$ occurs at

$$A = \{2m/5, 2m/5, ..., 2m/5\}, K = 2m/5 - 1$$
$$\text{or } A = \{3m/5, 3m/5, ..., 3m/5\}, K = 3m/5 - 1.$$

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