Evaluating a Determinant

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Outline of the talk

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- **First Part:** Introduction and background materials.

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- **Second Part:** Our works: variations of Zeilberger method and Krattenthaler's determinant conjectures.

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- It was Cauchy who used the word **determinant** in its present sense.
- **Note:** Inverse Matrix, Determinant and solving the system of linear equations are strongly tied together.

Definitions

- **Leibniz Formula**

  The determinant of an n-by-n matrix A is

  $$\det(A) = \sum_{\sigma \in S_n} (-1)^{\mathrm{Inv}(\sigma)} \Pi_{k=1}^{n} a_{k,\sigma_k}.$$

  where $\mathrm{Inv}(\sigma) :=$ number of pairs (i, j) in σ such that $i < j$ and $\sigma(i) > \sigma(j)$.

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  where $\mathrm{Inv}(\sigma) :=$ number of pairs (i,j) in σ such that $i < j$ and $\sigma(i) > \sigma(j)$.
- **Example** The determinant of a 2x2 matrix is
  $$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.$$

Definitions(continued)

- **Laplace Expansion**

  The determinant of an n-by-n matrix A is

  $$\begin{aligned} \det(A) &= \textstyle\sum_{j=1}^{n}(-1)^{i+j}a_{i,j}M_{i,j}. \\ &= \textstyle\sum_{i=1}^{n}(-1)^{i+j}a_{i,j}M_{i,j}. \end{aligned}$$

  where $M_{i,j}$:= the determinant of an $(n-1)$-by-$(n-1)$ matrix that results from A by removing the i-th row and the j-th column.

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  where $M_{i,j}$:= the determinant of an $(n-1)$-by-$(n-1)$ matrix that results from A by removing the i-th row and the j-th column.
- **Example** The determinant of a 3x3 matrix is
  $$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = (-1)^{(3+1)}a_{3,1}\begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} +$$
  $$(-1)^{(3+2)}a_{3,2}\begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{vmatrix} + (-1)^{(3+3)}a_{3,3}\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}.$$

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- However the naive calculation of determinant using Laplace expansion is extremely inefficient. The number of required operations is of order $n!$ for an n-by-n matrix.
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- Laplace Expansion:
  $\det(A) =$
  $(-1)^{n+1}a_{n,1}M_{n,1} + (-1)^{n+2}a_{n,2}M_{n,2} + ... + (-1)^{n+n}a_{n,n}M_{n,n}.$

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- $0 = (-1)^{n+1} a_{i,1} M_{n,1} + (-1)^{n+2} a_{i,2} M_{n,2} + ... + (-1)^{n+n} a_{i,n} M_{n,n}, 1 \leq i < n.$

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- With this set of $n - 1$ linear equations with n unknowns, we need only to solve this system of equations and calculate only one, instead of n, of the determinant of size $n - 1$.

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- **Example** $\det_{1\leq i,j\leq n}\left(\binom{i+j}{j}\right) = n+1.$
- **Example** Vandermonde determinant
  $$\begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \\ 1 & 1 & \dots & 1 \end{vmatrix} = \prod_{1\leq i<j\leq n}(x_i - x_j).$$
- Solution afterward.

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- Some Applications:
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- Some Applications:
  - Linear independence of vectors in R^n.
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  - Jacobian determinant to calculate the new volume after a linear mapping.
- In our case, the determinant is an answer to the counting problem, namely plane partitions.

Partitions

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- **Example** 7+4+2+1 is a partition of 14.
  which can be represent by Ferrers diagram as

$$\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
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- **Examples** $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5, p(5) = 7$.

Plane Partitions

- **Definition:** a plane partition of n is a two-dimensional array of nonnegative integers $n_{i,j}$ which are nonincreasing from left to right and top to bottom $\sum_{1 \leq i \leq k, 1 \leq j \leq l} n_{i,j} = n$..

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- **Definition:** $pp(n)$ is the number of plane partitions of n.
- **Examples**
  $pp(1) = 1, pp(2) = 3, pp(3) = 6, pp(4) = 13, pp(5) = 24.$

Generating Function for Plane Partitions

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- **Note:** The generating function for partitions: $$\sum_{n=0}^{\infty} p(n)q^n = \prod_{j=1}^{\infty} \frac{1}{1-q^j}.$$

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The generating function for plane partitions that are subsets of $\beta(r,s,t)$ *is given by*

$$\prod_{(i,j,k)\in\beta(r,s,t)} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}.$$

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- **Note:** By letting $r,s,t \to \infty$, we get the generating function for plane partitions.

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  The generating function for partitions into at most r parts , each less than or equal to s is given by

$$\begin{bmatrix} r+s \\ s \end{bmatrix}$$

where $\begin{bmatrix} r+s \\ s \end{bmatrix} := \dfrac{[r+s]_q!}{[r]_q![s]_q!}$
and $[r]_q! := 1(1+q)(1+q+q^2)...(1+q+q^2+..+q^{r-1})$.

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- **Cyclically symmetric plane partitions** is a plane partitions for which (i,j,k) is an element if and only if (j,k,i) and (k,i,j) are elements.
- **Totally symmetric plane partitions** is a plane partitions for which (i,j,k) is an element if and only each permutation of these vertices yields an element of the plane partition.

More Generating Functions

- Theorem

  The generating function for **symmetric plane partitions** *that are subsets of* $\beta(r, r, t)$ *is given by*

$$\prod_{\eta \in \beta(r,r,t)/S_2} \frac{1 - q^{|\eta|(1+ht(\eta))}}{1 - q^{|\eta| ht(\eta)}}.$$

  where η is an orbit and $|\eta|$ denotes the number of elements in η.
  $ht(i, j, k)$ is the height of an element (i, j, k) defined by $ht(i, j, k) := i + j + k - 2$.

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  - Or we express it as

$$\left(\prod_{i=1}^{r}\prod_{k=1}^{t}\frac{1-q^{1+2i+k-2}}{1-q^{2i+k-2}}\right)\left(\prod_{1\le i<j\le r}\prod_{k=1}^{t}\frac{1-q^{2(1+i+j+k-2)}}{1-q^{2(i+j+k-2)}}\right).$$

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  The generating function for **cyclically symmetric plane partitions** *that are subsets of* $\beta(r,r,r)$ *is given by*

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- Theorem

  The **orbit counting** *generating function for* **totally symmetric plane partitions** *that are subsets of* $\beta(r,r,r)$ *is given by*

$$\prod_{\eta \in \beta(r,r,r)/S_3} \frac{1 - q^{1+ht(\eta)}}{1 - q^{ht(\eta)}}.$$

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The number of plane partitions that fit inside $\beta(r,s,t)$ *is*

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- The generating function of symmetric partition (which is an analog of symmetric plane partition) is again not difficult to calculate.
- The generating function of self-conjugate partition into at most r parts, each less than or equal to r is given by ???

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- Theorem

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  - For the counting function, we let $q \to 1$.

  Theorem

  The number of plane partitions that fit inside $\beta(r,s,t)$ is

$$\det_{1\le i,j\le r}\left(\binom{t+s}{s-i+j}\right).$$

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- Dodgson Condensation Method.
- Roots method.
- Zeilberger method.

Condensation Method

- Let A_n be an n-by-n matrix
  Let $B_k(i, j)$ be the determinant of the k-by-k minor of A_n consisting of k contiguous rows and columns of A_n starting with row i and column j.
  In particular $\det(A_n) = B_n(1, 1)$.

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  In particular $\det(A_n) = B_n(1,1)$.

- Theorem
  (Desnanot-Jacobi adjoint matrix theorem)

$$B_n(1,1)B_{n-2}(2,2) =$$
$$B_{n-1}(1,1)B_{n-1}(2,2) - B_{n-1}(2,1)B_{n-1}(1,2).$$

Example 1 of Dodgson Condensation Method

- **Example:** The determinant of a 3x3 matrix can be calculated accordingly.
$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} a_{2,2} =$$
$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}.$$

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- **Example** $\det_{1\le i,j\le n}\left(\binom{i+j}{j}\right) = n+1.$
- **Step 1**, Make a more general conjecture.

$$\det_{1\le i,j\le n}\left(\binom{i+j+a+b-2}{j+a-1}\right) = \prod_{k=1}^{n}\prod_{j=1}^{b}\prod_{i=1}^{a}\frac{i+j+k-1}{i+j+k-2}.$$

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- **Step 1**, Make a more general conjecture.
  $$\det_{1\le i,j\le n}\left(\binom{i+j+a+b-2}{j+a-1}\right)=\prod_{k=1}^{n}\prod_{j=1}^{b}\prod_{i=1}^{a}\frac{i+j+k-1}{i+j+k-2}.$$
- We first simplify the formula on the right hand side for the future use.
  $\prod_{k=1}^{n}\prod_{j=1}^{b}\prod_{i=1}^{a}\frac{i+j+k-1}{i+j+k-2}$
  $=\prod_{k=1}^{n}\prod_{j=1}^{b}\frac{j+k+a-1}{j+k-1}$
  $=\prod_{k=1}^{n}\frac{(k+a+b-1)!}{(k+a-1)!}\frac{(k-1)!}{(k+b-1)!}$
  $=\frac{(n+a+b-1)!!}{(a+b-1)!!}(n-1)!!\frac{(a-1)!!}{(n+a-1)!!}\frac{(b-1)!!}{(n+b-1)!!}$
  where $a!!:=0!1!2!\ldots a!$.

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- Define $T_n(a,b) := \frac{(n+a+b-1)!!}{(a+b-1)!!}(n-1)!!\frac{(a-1)!!}{(n+a-1)!!}\frac{(b-1)!!}{(n+b-1)!!}$.

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- **Base case:** $n = 1$,
  LHS: $\det(\binom{i+j+a+b-2}{i+a-1})_{i,j=1}^{1} = \binom{a+b}{a}$.
  RHS: $T_1(a,b) = \frac{(1+a+b-1)!!}{(a+b-1)!!}(1-1)!!\frac{(a-1)!!}{(1+a-1)!!}\frac{(b-1)!!}{(1+b-1)!!}$
  $= \frac{(a+b)!}{a!b!}$

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- LHS: $\det(\binom{i+j+a+b-2}{i+a-1})_{i,j=1}^{2} = \begin{vmatrix} \binom{a+b}{a} & \binom{a+b+1}{a+1} \\ \binom{a+b+1}{a} & \binom{a+b+2}{a+1} \end{vmatrix}$
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  $= \frac{(a+b+1)!(a+b)!}{(a+1)!a!(b+1)!b!}.$
- RHS: $T_2(a,b) = \frac{(2+a+b-1)!!}{(a+b-1)!!}(2-1)!!\frac{(a-1)!!}{(2+a-1)!!}\frac{(b-1)!!}{(2+b-1)!!}$

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- We need the right hand side to satisfy the same relation. We show that:
  $T_n(a,b)T_{n-2}(a+1,b+1) =$
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  In this calculation, we use computer to verify the relation.
- Done! Hence, $\det\limits_{1\leq i,j\leq n}\left(\binom{i+j}{j}\right) = T_n(1,1) = n+1.$

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- **Con:**
  - More limited type of problems that this method can solve.

Roots Method

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- **Example** Vandermonde determinant

$$\begin{vmatrix} x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \dots & x_n^{n-2} \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_n \\ 1 & 1 & \dots & 1 \end{vmatrix} = \prod_{1 \le i < j \le n} (x_i - x_j).$$

Example of Roots Method

- First, we notice that by replacing x_i by x_j, $i \neq j$, the determinant of this new matrix becomes 0. Hence $(x_i - x_j)$ is a factor of this determinant.

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- Second, we proceed using induction on n, the number of variables.

  Base Case: We verify the statement is true for $n = 2$.

  Induction Step: Assume the statement is true for all $k < n$.

  We proceed by extracting $\displaystyle\prod_{j=2}^{n}(x_1 - x_j)$ from $\det(A_n)$.

  $$\det(A_n) = g(x_1, x_2, ..., x_n)\prod_{j=2}^{n}(x_1 - x_j).$$

Example of Roots Method(Continued)

- By comparing the coefficient of x_1^{n-1} on both sides, we have

$$g(x_1, x_2, ..., x_n) = \begin{vmatrix} x_2^{n-2} & x_3^{n-2} & ... & x_n^{n-2} \\ ... & ... & ... & ... \\ x_2 & x_3 & ... & x_n \\ 1 & 1 & ... & 1 \end{vmatrix}$$

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- We then conclude the result by claiming an induction.

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- **Pros:**
  - The main idea is simple.
  - Works well when the matrix has mixed types of entries.
- **Con:**
  - There is more difficulty when the determinant has repeated roots.
  - Nobody finds the way to computerize it yet. And it takes a long time to do by hand.

Zeilberger Method

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- Let $a_{i,j}$ be an entry (i,j) of matrix A of size n.
  By definition of Laplace expansion
  $$\det_{1\le i,j\le n}(a_{i,j}) = \sum_{j=1}^{n}(-1)^{n+j}a_{n,j}M_{n,j}.$$
  Let b_n be $\det\limits_{1\le i,j\le n}(a_{i,j})$.
  Assume $b_n \neq 0$ for all n.
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  Assume $b_n \neq 0$ for all n.
  Note: $M_{n,n} = b_{n-1}$.
- Define $c_{n,j} := (-1)^{n+j}M_{n,j}/M_{n,n}$.
  We have $b_n = \sum_{j=1}^{n} c_{n,j}a_{n,j}M_{n,n}$.

Zeilberger Method(Continued)

- Hence it follows that:

$$c_{n,n} = 1. \tag{1}$$

$$\sum_{j=1}^{n} c_{n,j} a_{i,j} = 0, \quad 1 \leq i < n. \tag{2}$$

$$\sum_{j=1}^{n} c_{n,j} a_{n,j} = \frac{b_n}{b_{n-1}}. \tag{3}$$

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  for $n = 1, 2, 3, \ldots.$
- Then we guess the closed-form formula or guess the linear recurrence relations with polynomial coefficients of $c_{n,j}$.

Computation of Zeilberger Method(Continued)

- If $c_{n,j}$ has a hypergeometric closed-form formula and $a_{i,j}$ is hypergeometric , we evaluate the sum in (2) and (3) by using Zeilberger's Algorithm.

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- If $c_{n,j}$ is not holonomic then we do not have an algorithm to verify (1),(2) and (3) yet.
- **Remark:** However before starting to solve any problem, it is always a good idea to check first that $\frac{b_n}{b_{n-1}}$ is holonomic.

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- Find the orbit counting generating function for **totally symmetric plane partitions** that are subsets of $\beta(r,r,r)$.
- **Remark:** By setting $q \to 1$, we change the generating function into the counting function.

Zeilberger Method and Plane Partitions (Continued)

- The last survived conjecture was the orbit counting generating function for **totally symmetric plane partitions** that are subsets of $\beta(r, r, r)$. This problem, just like other, was translated into determinant problem. This notoriously difficult problem was solved using this Zeilberger method just last year (2010) by Manuel Kauers, Christoph Koutschan and Doron Zeilberger.

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- Roles:
  Doron Zeilberger $\rightarrow$ Zeilberger method
  Manuel Kauers $\rightarrow$ Guess.m
  Christoph Koutschan $\rightarrow$ HolonomicFunctions.m

Zeilberger Method and Plane Partitions (Continued)

- Zeilberger method has been known since 2007 but the amount of calculation was too big in practice. However with a lot of tricks and optimizations from Manuel Kauers and Christoph Koutschan, the programs were efficient enough to handle the problem.

Part II, Variations of Zeilberger Method and Krattenthaler's Conjectures

- Prof. Krattenthaler made six conjectures about determinants in Advanced Determinant Calculus: a Complement(2005). Koutschan and I were able to solve three of them. All of them could not be solved straight from Zeilberger method. Some modifications of Zeilberger method are needed.

Krattenthaler's Conjecture 1

- Theorem

  Let the determinant $D_1(n)$ be defined by

$$D_1(n) := \det_{1 \le i,j \le n} \left(\delta_{i,j} + \binom{\mu + i + j - 2}{j} \right)$$

where μ is an indeterminate. Then the following relation holds:

$$\frac{D_1(2n)}{D_1(2n-1)} = (-1)^{\binom{n-1}{2}} \frac{2^n \left(\frac{\mu}{2} + n\right)_{\lceil n/2 \rceil} \left(\frac{\mu}{2} + 2n + \frac{1}{2}\right)_{n-1}}{(n)_n \left(-\frac{\mu}{2} - 2n + \frac{3}{2}\right)_{\lceil (n-2)/2 \rceil}},$$

where $a_k := a(a+1)(a+2)...(a+k-1)$.

Solution to Krattenthaler's Conjecture 1

- We notice that the patterns occur at $\frac{D_1(2n)}{D_1(2n-1)}$ but not $\frac{D_1(2n+1)}{D_1(2n)}$.

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- We notice that the patterns occur at $\frac{D_1(2n)}{D_1(2n-1)}$ but not $\frac{D_1(2n+1)}{D_1(2n)}$.
- Therefore we need to switch our attention to only an even n. We guess the recurrences for $c_{i,j}$ only for a matrix of even size.

Zeilberger Method, First Variation

- The new identities, to be verified, from Zeilberger method will be

$$c_{n,2n} = 1.$$

$$\sum_{j=1}^{2n} c_{n,j} a_{i,j} = 0, \quad 1 \leq i < 2n.$$

$$\sum_{j=1}^{2n} c_{n,j} a_{2n,j} = \frac{D_1(2n)}{D_1(2n-1)}.$$

Application for Determinant of a Skew Symmetric Matrix

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- **Recall:** Desnanot-Jacobi adjoint matrix theorem:
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- **Note:** $b_n(1,1) = \det(A_n)$.

Application for determinant of Skew Symmetric Matrix

- We substitute $n \to 2n+2$ and $n \to 2n+1$ in Desnanot-Jacobi adjoint matrix theorem then apply $b_{2n+1}(1,1) = 0$:

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- After dividing the two equations, we have $\dfrac{b_{2n+2}(1,1)}{b_{2n}(1,1)} = -\dfrac{b_{2n+1}(1,2)}{b_{2n}(1,2)}\dfrac{b_{2n+1}(2,1)}{b_{2n}(2,1)}$.

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- Since $-b_{2n+1}(2,1) = b_{2n+1}(1,2)$ and $b_{2n}(2,1) = b_{2n}(1,2)$, We have

$$\frac{b_{2n+2}(1,1)}{b_{2n}(1,1)} = \left(\frac{b_{2n+1}(1,2)}{b_{2n}(1,2)}\right)^2.$$

Zeng's Problem

- Theorem

$$\det_{1\le i,j\le 2n}\left((j-i)M_{i+j-3}\right)=\prod_{k=1}^{n}(4k-3)^2$$

where M_n denotes Motzkin number defined by

$$M_n := \sum_{k=0}^{n}\binom{n}{2k}C_k$$

where C_k denote Catalan number defined by $C_k := \frac{1}{k+1}\binom{2k}{k}$. M_n also satisfies the recurrence

$$(n+4)M_{n+2}=(2n+5)M_{n+1}+(3n+3)M_n$$

where $M_{-1}=0$ and $M_0=1$.

Krattenthaler's Conjecture 2

- Theorem

  Let μ be an indeterminate and n be a non-negative integer. The determinant

$$\det_{1\le i,j\le n}\left(-\delta_{i,j}+\binom{\mu+i+j-2}{j}\right)$$
$$=(-1)^{n/2}2^{n(n+2)/4}\frac{\left(\frac{\mu}{2}\right)_{n/2}}{\left(\frac{n}{2}\right)!}\left(\prod_{i=0}^{(n-2)/2}\frac{i!^2}{(2i)!^2}\right)\times$$
$$\left(\prod_{i=0}^{\lfloor (n-4)/4\rfloor}\left(\frac{\mu}{2}+3i+\frac{5}{2}\right)_{(n-4i-2)/2}^2\left(-\frac{\mu}{2}-\frac{3n}{2}+3i+3\right)_{(n-4i-4)/2}^2\right)$$

  if n is even,

Krattenthaler's Conjecture 2(Continued)

- $$= (-1)^{(n-1)/2} 2^{(n+3)(n+1)/4} \left(\frac{\mu-1}{2}\right)_{(n+1)/2} \left(\prod_{i=0}^{(n-1)/2} \frac{i!(i+1)!}{(2i)!(2i+2)!}\right) \times \left(\prod_{i=0}^{\lfloor (n-3)/4 \rfloor} \left(\frac{\mu}{2} + 3i + \frac{5}{2}\right)^2_{(n-4i-3)/2}\right) \times \left(\prod_{i=0}^{\lfloor (n-3)/4 \rfloor} \left(-\frac{\mu}{2} - \frac{3n}{2} + 3i + \frac{3}{2}\right)^2_{(n-4i-1)/2}\right)$$

if n is odd.

Solution to Krattenthaler's Conjecture 2

- In theory, by Zeilberger method, first variation, we can compute $\dfrac{\det(A_{2n})}{\det(A_{2n-1})}$ and $\dfrac{\det(A_{2n+1})}{\det(A_{2n})}$ separately.

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- However in practice, the computation was too big. We need to go around to find a way to break this determinant into some smaller pieces that are smaller to compute in parts.
- We will do this problem in 2 steps.
  Step 1: we compute six different determinant and ratios of determinants.
  Step 2: we apply adjoint matrix theorem to conclude the theorem.

Solution to Krattenthaler's Conjecture 2

- Let $b_n(I, J) :=$
$$\det_{I \le i \le n-1+I, J \le j \le n-1+J} \left(-\delta_{i,j} + \binom{\mu + i + j - 2}{j} \right).$$
We want to find $b_n(1, 1)$ when n is odd and even.

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We want to find $b_n(1,1)$ when n is odd and even.
- **Step1:**
Show the formulas of the following by using Zeilberger method

$$b_{2n-1}(0,0)=0. \quad \frac{b_{2n}(0,0)}{b_{2n-1}(1,1)}=-1.$$
$$\frac{b_{2n}(1,0)}{b_{2n-1}(1,0)}=Nice1. \quad \frac{b_{2n}(0,1)}{b_{2n-1}(0,1)}=Nice2.$$
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- **Note:** In this problem, we are lucky enough that the dimension of null space, for each n, is 1. There is no difficulty in guessing the recurrences of $c_{n,j}$.

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- The conditions that we have to verify are

$$c_{n,0} = 1.$$

$$\sum_{j=0}^{2n-1} c_{n,j} a_{i,j} = 0, \quad 0 < i \le 2n-1.$$

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Last four equations in Step 1

- **To Show:**

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$$\frac{b_{2n+1}(1,0)}{b_{2n}(1,0)} = Nice3. \quad \frac{b_{2n+1}(0,1)}{b_{2n}(0,1)} = Nice4.$$
- Since these four equations are the ratio of the determinant with the same starting point, we can handle them the same way we did in Krattenthaler's conjecture 1 by using Zeilberger method, first variation.

Solution to Krattenthaler's Conjecture 2: Step2

- **Step2:**
  We show $\dfrac{b_{2n+1}(1,1)}{b_{2n-1}(1,1)}$ and $\dfrac{b_{2n}(1,1)}{b_{2n-2}(1,1)}$ using results from step 1 and Desnanot-Jacobi adjoint matrix theorem,

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- Plug $n \to 2n+2$, $n \to 2n+1$ and $n \to 2n$ in adjoint matrix theorem then apply $b_{2n-1}(0,0) = 0$ to get
  $$b_{2n+2}(0,0)b_{2n}(1,1) = -b_{2n+1}(0,1)b_{2n+1}(1,0) \quad (1)$$
  $$b_{2n}(0,0)b_{2n}(1,1) = b_{2n}(0,1)b_{2n}(1,0) \quad (2)$$
  $$b_{2n}(0,0)b_{2n-2}(1,1) = -b_{2n-1}(0,1)b_{2n-1}(1,0) \quad (3)$$

Solution to Krattenthaler's Conjecture 2: Step2(Continued)

- Next, we do $(1)/(2)$ and $(2)/(3)$ then apply $\dfrac{b_{2n}(0,0)}{b_{2n-1}(1,1)} = -1$ to get

$$\frac{b_{2n+1}(1,1)}{b_{2n-1}(1,1)} = -\frac{b_{2n+1}(1,0)}{b_{2n}(1,0)}\frac{b_{2n+1}(0,1)}{b_{2n}(0,1)}.$$
$$\frac{b_{2n}(1,1)}{b_{2n-2}(1,1)} = -\frac{b_{2n}(1,0)}{b_{2n-1}(1,0)}\frac{b_{2n}(0,1)}{b_{2n-1}(0,1)}.$$

Krattenthaler's Conjecture 3

- Theorem

  Let μ be an indeterminate. For any odd non-negative integer n there holds

$$\det_{1\le i,j\le n}\left(-\delta_{i,j}+\binom{\mu+i+j-2}{j+1}\right)$$

$$=(-1)^{(n-1)/2}2^{(n-1)(n+5)/4}(\mu+1)\frac{\left(\frac{\mu}{2}-1\right)_{(n+1)/2}}{\left(\frac{n+1}{2}\right)!}$$

$$\times\left(\prod_{i=0}^{(n-1)/2}\frac{i!^2}{(2i)!^2}\prod_{i=0}^{\lfloor (n-1)/4\rfloor}\left(\frac{\mu}{2}+3i+\frac{3}{2}\right)^2_{(n-4i-1)/2}\right)$$

$$\times\left(\prod_{i=0}^{\lfloor (n-3)/4\rfloor}\left(-\frac{\mu}{2}-\frac{3n}{2}+3i+\frac{5}{2}\right)^2_{(n-4i-3)/2}\right).$$

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- Therefore we apply Zeilberger method, third variation, for the formula of $\dfrac{b_{2n}(1,1,\mu-2)}{b_{2n-1}(2,2,\mu-2)}$.
- Since $b_{2n}(1,1,\mu-2)$ is known from Krattenthaler's Conjecture 2. We are done.

Zeilberger Method, Fourth Variation for b_n/b_{n-K}

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- This variation of Zeilberger method came from the block matrix identity which states as following:

$$\det(M) = \det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_1)\det(M_4 - M_3 M_1^{-1} M_2).$$

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- This variation is powerful. However the drawback is the bigger calculation we have to deal with. From examples that we faced so far, there is always a way to use other variation with smaller computation.