Generalized Fibonacci Numbers with Matrix Method

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Introduction to Matrix Form

We unified all the generalized Fibonacci numbers in the past using the matrix form. This way we have a simple way to show the known identities using the properties of the matrix. We also find the new identities by this method.

Fibonacci numbers

We define Fibonacci numbers, F_n, by the following matrix:

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n .$$

Known Identities

Many of the known identities can be proved by this matrix method. For example:

$$F_{n+1}F_{m+1} + F_nF_m = F_{n+m+1}.$$

Proof.

Consider the top right entry of the following:

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \cdot \begin{bmatrix} F_{m+2} & F_{m+1} \\ F_{m+1} & F_m \end{bmatrix} = P^n P^{m+1} = P^{n+m+1}$$

$$= \begin{bmatrix} F_{n+m+2} & F_{n+m+1} \\ F_{n+m+1} & F_{n+m} \end{bmatrix},$$

where $P := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. $\square$

Known identites

Second example:

$$\sum_{j=1}^{n} F_j = F_{n+2} - 1$$

Proof.

Comparing the top right entry of the following:

$\sum_{j=1}^{n} P^j = \dfrac{P(I-P^n)}{I-P} = \dfrac{P(I-P^n)}{-P^{-1}} = P^{n+2} - P^2.$ □

Generalizations, Lucas numbers

$$\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Noticed that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

gives rise to many nice identities.

Gaussian Fibonacci

Gaussian Fibonacci mentioned by Jordan [FQ,1965],

$$GF_n = F_n + iF_{n-1}$$

can be defined by matrix method as following:

$$\begin{bmatrix} GF_{n+1} & GF_n \\ GF_n & GF_{n-1} \end{bmatrix} := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix}.$$

Gaussian Fibonacci (continued)

Remark:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix}^2 = \begin{bmatrix} 1+2i & 0 \\ 0 & 1+2i \end{bmatrix}$$

gives rise to some nice identity such as:

$$GF_n(GF_{n+1} + GF_{n-1}) = (1+2i)F_{2n-1}$$

Proof.

Consider the top right entry of the following:

$$\begin{bmatrix} GF_{n+1} & GF_n \\ GF_n & GF_{n-1} \end{bmatrix} \cdot \begin{bmatrix} GF_{n+1} & GF_n \\ GF_n & GF_{n-1} \end{bmatrix} = (P^n R)^2 = P^{2n-1}(PR^2).$$

Fibonacci Quaternions

Fibonacci Quaternions mentioned by Iyer [FQ,1969],

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$$

can be defined by matrix method as following:

$$\begin{bmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{bmatrix}.$$

Fibonacci Quaternions(continued)

Matrix method gives a simple prove to these identities

1. $Q_n - iQ_{n+1} - jQ_{n+2} - kQ_{n+3} = 3L_{n+3}$.
2. $Q_n^2 + Q_{n-1}^2 = 2Q_{2n-1} - 3L_{2n+2}$.
3. $Q_{n+1}^2 + Q_{n-1}^2 = 6F_{n+1}Q_{n-1} - 9F_{2n+3} + 2(-1)^{n+1}(1 - i - k)$.
4. $\sum_{i=0}^{n} Q_i = Q_{n+2} - Q_1$.

Gaussian Fibonacci numbers by Berzsenyi

Berzsenyi, [FQ,1977], gave the definition of $F_{n,m} := F_{n+mi}$ that satisfies the discrete version of Cauchy-Riemann equation

$$\frac{f(z+i)-f(z)}{i} = f(z+1)-f(z)$$

$$\rightarrow \frac{F_{n,m+1}-F_{n,m}}{i} = F_{n+1,m}-F_{n,m}.$$

This leads us to the beautiful formula of $F_{n,m}$

$$F_{n,m} = \sum_{k=0}^{m} \binom{m}{k} i^k F_{n-k}.$$

Gaussian Fibonacci numbers by Berzsenyi (continued)

Remark: $F_{n,0} = F_n$ and $F_{n,1} = F_n + iF_{n-1} = GF_n$.
Gaussian Fibonacci numbers of Berzsenyi can be defined as following:

$$\begin{bmatrix} F_{n+1,m} & F_{n,m} \\ F_{n,m} & F_{n-1,m} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix}^m .$$

Known identity

$$\sum_{j=0}^{2m} \binom{2m}{j} i^j F_{n-j} = (1+2i)^m F_{n-m}$$

can be obtained by the matrix method thanks to the fact that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix}^2 = \begin{bmatrix} 1+2i & 0 \\ 0 & 1+2i \end{bmatrix}.$$

What else?

We define the new sequence $T_{n,m}$ as following:

$$\begin{bmatrix} T_{n+1,m} & T_{n,m} \\ T_{n,m} & T_{n-1,m} \end{bmatrix} := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 & b \\ b & 1-b \end{bmatrix}^m .$$

Main Theorem

Theorem

If there is an integers A, B and a complex number b satisfy the following:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^A \cdot \begin{bmatrix} 1 & b \\ b & 1-b \end{bmatrix}^B = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

then

$$c^m F_n = \sum_j \binom{Bm}{j} b^j F_{n+Am-j}, \text{ for any } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}.$$

Some identities from the Theorem

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^0 \cdot \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

gives

$$5^m F_n = \sum_j \binom{2m}{j} 2^j F_{n-j}.$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^1 \cdot \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix}^2 = \begin{bmatrix} 1+2i & 0 \\ 0 & 1+2i \end{bmatrix}$$

gives

$$(1+2i)^m F_n = \sum_j \binom{2m}{j} i^j F_{n+m-j}.$$

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Table

List of A, B and b such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^A \cdot \begin{bmatrix} 1 & b \\ b & 1-b \end{bmatrix}^B = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

A	B	b	c	Note
0	1	0	1	Identity
0	2	2	5	Lucas
1	2	$\pm i$	$1 \pm 2i$	Complex Fibo
2	1	-1	1	
2	2	$\frac{-1}{2}$	$\frac{5}{4}$	
2	4	$2 \pm \sqrt{5}$	$25(9 \pm 4\sqrt{5})$	
3	1	-2	-1	
3	2	$-1 \pm i$	$-1 \pm 2i$	
4	1	$-\frac{3}{2}$	$\frac{1}{2}$	
...				