Fibonacci Identities through Matrix Method

Thotsaporn "Aek" Thanatipanonda
Mahidol University International College

December 25, 2018

1 Introduction

We unified all the generalized Fibonacci numbers in the past by using the matrix method. This method is by no mean new, it has been used extensive by many author and was a chapter in the book *Fibonacci and Lucas Number with Applications* by Koshy, [11]. However, many papers did not use this matrix method to show their identities. It is common that we have a simpler way to show the old identities using the properties of the matrix. We also successfully find the new generalization of the numbers in the past. The outline of the paper are:

Section 2: Apply the matrix method to the known identities.

Section 2.1: On Fibonacci identities from couple papers, Harman [7], Demirturk and Kesking [4], Benjamin and Quinn [1] and Bloom [3] .

Section 2.2: On identities of Gaussian Fibonacci numbers

$$GF_n = GF_{n-1} + GF_{n-2},$$

where $GF_0 = a + bi$ and $GF_1 = c + di$ from Jordon [10] .

Section 2.3: On identities of Fibonacci Quaternions

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$$

by Iyer [8] and Halici [6].

Section 3: We consider the matrix of the form

$$\begin{bmatrix} T_{n+1,m} & T_{n,m} \\ T_{n,m} & T_{n-1,m} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a & b \\ b & a-b \end{bmatrix}^m.$$

Case $a = 1, b = i$ is the work of Berzsenyi, *Gaussian Fibonacci Numbers*, [2].
Case $a = 1, b = 2$ is the generalization of Iyer, [8], we mentioned in section 2.
We investigated the results of the general form in section 3.3 and 3.4.

Section 4: We deal with the numbers arise from the more general form of matrix multiplication, i.e.

$$\begin{bmatrix} T_{n+1,m,p} & T_{n,m,p} \\ T_{n,m,p} & T_{n-1,m,p} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a & b \\ b & a-b \end{bmatrix}^m \begin{bmatrix} A & B \\ B & A-B \end{bmatrix}^p.$$

Section 5: We consider the numbers that arise from multiplication of the two 3-by-3 matrices, i.e. tribonacci numbers, trucas numbers. We also consider the identities from general form in section 5.3.

Section 6: Another direction of the generalization of Complex Fibonacci numbers by Harman, [7]:

$$G(n+2,m) = G(n+1,m) + G(n,m),$$

$$G(n,m+2) = G(n,m+1) + G(n,m),$$

where $G(0,0) = 0, \quad G(1,0) = 1, \quad G(0,1) = i, \quad G(1,1) = 1+i.$

Then we generalize this idea.

2 One Matrix Multiply, $P^n R$

We will use the matrix to define several known generalized Fibonacci numbers and prove their identities.

In this section, let a 2-by-2 matrix P and R be in the form:

$P := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $R := \begin{bmatrix} a & b \\ b & a-b \end{bmatrix}$.

We consider the matrix of the form $P^n R$.
Note that $P^n = F_n P + F_{n-1} I$, i.e. $P^2 = P + I$.

2.1 Fibonacci numbers

We define Fibonacci numbers, F_n, by the following matrix:

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

Note that the matrix is well defined.

2.1.1 Identities

The matrix form gives rise to the more compact form of "many" known identities.

1. $F_2 = 1, F_1 = 1$ and $F_0 = 0$. By substitute $n = 1$ in the definition.

2. $F_{n+2} = F_{n+1} + F_n$.
   Using the fact that $P^2 = P + I$ then read off from the entry in of matrix $P^{n+2} = P^{n+1} + P^n$.

3. $\sum_{j=1}^{n} F_j = F_{n+2} - 1$.

   By the fact that $\sum_{j=1}^{n} P^j = \dfrac{P(I - P^n)}{I - P} = \dfrac{P(I - P^n)}{-P^{-1}} = P^{n+2} - P^2$.

4. $\sum_{j=1}^{n} F_{2j-1} = F_{2n}$.

   By the fact that $\sum_{j=1}^{n} P^{2j-1} = \dfrac{P(I - P^{2n})}{I - P^2} = P^{2n} - I$.
   i.e. $F_7 + F_5 + F_3 + F_1 = F_8$. From the matrix identity $P^7 + P^5 + P^3 + P + I = P^8$.

5. $\sum_{j=1}^{n} F_{4j} = \dfrac{3F_{4n+5} - 4F_{4n+4} - 3}{5}$. Proof can be done in similar fashion.

6. $F_{n+1}F_{m+1} + F_nF_m = F_{n+m+1}$.

   Consider the top right entry of the following:

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \cdot \begin{bmatrix} F_{m+2} & F_{m+1} \\ F_{m+1} & F_m \end{bmatrix} = P^n P^{m+1} = P^{n+m+1}.$$

   Special case $m = n - 1$:

   $F_{2n} = F_{n+1}F_n + F_nF_{n-1} = F_n(F_{n+1} + F_{n-1}) = F_{n+1}^2 - F_{n-1}^2$.

7. $\sum_{j=1}^{k} F_{4j-2} = F_{2k}^2$ and $\sum_{j=1}^{k} F_{4j} = F_{2k+1}^2$

   This can be shown by substituting $n = 2j - 1$ and $n = 2j$ respectively in the previous result then do the telescoping sum.

8. (Eq 3.6 of Harman, [7])

$$F_{n+2k+1}F_{m+2k} - F_{n+2k}F_{m+2k+1} = F_{n+1}F_m - F_nF_{m+1}.$$

   This can be shown by using the definition

$$\begin{bmatrix} F_{n+2k+1} & F_{m+2k+1} \\ F_{n+2k} & F_{m+2k} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{2k} \cdot \begin{bmatrix} F_{n+1} & F_{m+1} \\ F_n & F_m \end{bmatrix}$$

   Then take the determinant both sides.

9. (Theorem 6 of Demirturk and Kesking, [4])

$$F_{n+k}F_{m+k} - F_kF_{n+m+k} = (-1)^k F_nF_m,$$

This can be shown by using the definition

$$\begin{aligned}
\begin{bmatrix} F_{n+k} & F_k \\ F_{n+m+k} & F_{m+k} \end{bmatrix} &= \begin{bmatrix} F_{k+1} & F_k \\ F_{m+k+1} & F_{m+k} \end{bmatrix} \cdot \begin{bmatrix} F_n & 0 \\ F_{n-1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ F_{m+1} & F_m \end{bmatrix} \cdot \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \cdot \begin{bmatrix} F_n & 0 \\ F_{n-1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ F_{m+1} & F_m \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \cdot \begin{bmatrix} F_n & 0 \\ F_{n-1} & 1 \end{bmatrix}
\end{aligned}$$

Then take the determinant both sides.

10. (Theorem 6 of Demirturk and Kesking, [4])

$$F_{n+k}F_m - F_kF_{n+m} = (-1)^k F_nF_{m-k},$$

Substitute m by $m-k$ in the previous result.

11. (Theorem 5.1 of Demirturk and Kesking, [4])

$$L_{n+m+k}L_k - L_{n+k}L_{m+k} = 5(-1)^k F_nF_m,$$

This can be shown by using the definition

$$\begin{aligned}
\begin{bmatrix} L_{n+m+k} & L_{n+k} \\ L_{m+k} & L_k \end{bmatrix} &= \begin{bmatrix} L_{n+k+1} & L_{n+k} \\ L_{k+1} & L_k \end{bmatrix} \cdot \begin{bmatrix} F_m & 0 \\ F_{m-1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} F_{n+1} & F_n \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} L_{k+1} & L_k \\ L_k & L_{k-1} \end{bmatrix} \cdot \begin{bmatrix} F_m & 0 \\ F_{m-1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} F_{n+1} & F_n \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} F_m & 0 \\ F_{m-1} & 1 \end{bmatrix}
\end{aligned}$$

Then take the determinant both sides.

12. (Theorem 5.2 of Demirturk and Kesking, [4]. substitute m with $m+k$)

$$F_{n+m+k}L_k - L_{n+k}F_{m+k} = (-1)^k L_mF_n,$$

This can be shown by using the definition

$$\begin{aligned}
\begin{bmatrix} F_{n+m+k} & L_{n+k} \\ F_{m+k} & L_k \end{bmatrix} &= \begin{bmatrix} F_n & F_{n-1} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} F_{m+k+1} & L_{k+1} \\ F_{m+k} & L_k \end{bmatrix} \\
&= \begin{bmatrix} F_n & F_{n-1} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} F_{m+k+1} & F_{m+k} \\ F_{m+k} & F_{m+k-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & L_{-m+1} \\ 0 & L_{-m} \end{bmatrix} \\
&= \begin{bmatrix} F_n & F_{n-1} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{m+k} \cdot \begin{bmatrix} 1 & L_{-m+1} \\ 0 & L_{-m} \end{bmatrix}.
\end{aligned}$$

Then take the determinant both sides.

13. (Finite sum, Theorem 2.1.2 of Demirturk, [5])

$$\sum_{j=0}^{n} F_{mj+k} = \frac{F_k - F_{mn+m+k} + (-1)^m (F_{mn+k} - F_{k-m})}{1 + (-1)^m - L_m}.$$

Consider the (2,1) entry of $P^k + P^{m+k} + P^{2m+k} + \cdots + P^{mn+k}$:

$$\begin{aligned} P^k + P^{m+k} + P^{2m+k} + \cdots + P^{mn+k} &= \frac{P^k (P^{(n+1)m} - I)}{P^m - I} \\ &= \frac{P^k (P^{(n+1)m} - I)((-1)^m P^{-m} - I)}{(F_{m+1}F_{m-1} - (F_{m+1} + F_{m-1}) + 1 - F_m^2)} \\ &= \frac{P^k - P^{nm+m+k} + (-1)^m (P^{nm+k} - P^{k-m})}{(-1)^m + 1 - L_m}. \end{aligned}$$

14. (Finite sum, Theorem 2.1.1 of Demirturk, [5])

$$\sum_{j=0}^{n} L_{mj+k} = \frac{L_k - L_{mn+m+k} + (-1)^m (L_{mn+k} - L_{k-m})}{1 + (-1)^m - L_m}.$$

The proof is similar to the previous.

15. (Binomial Sum, Lemma 2.1 of Demirturk and Kesking, [4])

$$\sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} F_{j+k} = (-1)^{k+1} \sum_{j=0}^{n} \binom{n}{j} (-a)^j (a+b)^{n-j} F_{j-k}.$$

Consider the upper right hand corner of $P^k(aP + bI)^n$:

On one hand: $P^k(aP + bI)^n = \sum_j \binom{n}{j} a^j b^{n-j} P^{j+k}$.

On the other hand:

$$P^k(aP + bI)^n = P^k((a+b)I + aP^{-1})^n = \sum_j \binom{n}{j} a^j (a+b)^{n-j} P^{-j+k}.$$

16. (Eq 3.8 of Harman, [7])

$$\sum_{j=1}^{2k} F_{n+j} F_{m+j} = F_{n+2k+1} F_{m+2k} - F_{n+1} F_m.$$

We can show this by induction on k.
The base case when $k = 0$ is true.
Induction step: let $S(k) = \sum_{j=1}^{2k} F_{n+j} F_{m+j}$.

$$\begin{aligned} S(k) - S(k-1) &= F_{n+2k} F_{m+2k} + F_{n+2k-1} F_{m+2k-1} \\ &= (F_{n+2k+1} - F_{n+2k-1}) F_{m+2k} + F_{n+2k-1} (F_{m+2k} - F_{m+2k-2}) \\ &= F_{n+2k+1} F_{m+2k} - F_{n+2k-1} F_{m+2k-2}. \end{aligned}$$

17. (Eq 3.9 of Harman, [7])

$$\sum_{j=1}^{2k+1} F_{n+j}F_{m+j} = F_{n+2k+2}F_{m+2k+1} - F_nF_{m+1}$$

This could be done similarly to 3.8.

18. (Eq 3.7 of Harman, [7])

$$\sum_{j=1}^{2k} F_{n+j}F_{m+j} = \frac{1}{2}\left(F_{n+2k}F_{m+2k+1} + F_{n+2k+1}F_{m+2k} - F_{n+1}F_m - F_nF_{m+1}\right).$$

This could also be done similarly to 3.8.

2.1.2 Sum of Products of Fibonacci

This section we consider $\sum_{i+j=n} F_iF_j, \quad \sum_{i+j+k=n} F_iF_jF_k, \ldots$ which was asked by Bloom, [3], in 1996.

Definition. Let $h_{i,j}$ be defined as
$h_{0,j} = 0$ if $j \geq 0$.
$h_{j,j} = 1$ if $j \geq 1$.
$h_{i,j} = 1$ if $i > j$.
$h_{i,j} = h_{i,j-1} + h_{i,j-2} + h_{i-1,j-1}$ if $i \geq 1$ and $j \geq 2$.

Theorem 2.1.1.

$$h_{1,n} = F_n.$$

Proof. It is obvious from the definition. □

Theorem 2.1.2.

$$h_{2,n} = \sum_{\substack{i,j>0 \\ i+j=n}} F_iF_j.$$

Proof. Let $H_{2,n} := \begin{bmatrix} h_{2,n+1} & h_{2,n} \\ h_{2,n} & h_{2,n-1} \end{bmatrix}$ and $R := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

By definition of $h_{i,j}$, we have that

$$\begin{aligned} H_{2,n+1} &= H_{2,n}P + P^nR \\ &= (H_{2,n-1}P + P^{n-1}R)P + P^nR \\ &= H_{2,n-1}P^2 + P^{n-1}RP + P^nR \\ &= H_{2,n-2}P^3 + P^{n-2}RP^2 + P^{n-1}RP + P^nR \\ &= \cdots \\ &= \sum_{k=0}^{n} P^{n-k}RP^k. \end{aligned}$$

Then we note that

$$\begin{bmatrix} F_{n-k+1} & F_{n-k} \\ F_{n-k} & F_{n-k-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} = \begin{bmatrix} F_{n-k+1} & 0 \\ F_{n-k} & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} = \begin{bmatrix} F_{n-k+1}F_{k+1} & F_{n-k+1}F_k \\ F_{n-k}F_{k+1} & F_{n-k}F_k \end{bmatrix}.$$

Therefore

$$\sum_{k=0}^{n} P^{n-k}RP^k = \begin{bmatrix} \sum_{k=0}^{n} F_{n+1-k}F_{k+1} & \sum_{k=0}^{n} F_{n+1-k}F_k \\ \sum_{k=0}^{n} F_{n-k}F_{k+1} & \sum_{k=0}^{n} F_{n-k}F_k \end{bmatrix}$$

□

Theorem 2.1.3.

$$h_{3,n} = \sum_{\substack{i,j,k>0 \\ i+j+k=n}} F_iF_jF_k.$$

Proof. Similar to the above theorem

$$\begin{aligned} H_{3,n+1} &= H_{3,n}P + H_{2,n}R \\ &= (H_{3,n-1}P + H_{2,n-1}R)P + H_{2,n}R \\ &= H_{3,n-1}P^2 + H_{2,n-1}RP + H_{2,n}R \\ &= H_{3,n-2}P^3 + H_{2,n-2}RP^2 + H_{2,n-1}RP + H_{2,n}R \\ &= \cdots \\ &= \sum_{k=0}^{n} H_{2,n-k}RP^k. \end{aligned}$$

Then note that

$$\begin{aligned} H_{2,n-k}RP^k &= \begin{bmatrix} \sum_{j=0}^{n-k} F_{n-k-j}F_{j+1} & \sum_{j=0}^{n-k} F_{n-k-j}F_j \\ \sum_{j=0}^{n-k} F_{n-1-k-j}F_{j+1} & \sum_{j=0}^{n-k} F_{n-1-k-j}F_j \end{bmatrix} \begin{bmatrix} F_{k+1} & F_k \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^{n+1-k} F_{n+1-k-j}F_jF_{k+1} & \sum_{j=1}^{n+1-k} F_{n+1-k-j}F_jF_k \\ \sum_{j=1}^{n+1-k} F_{n-k-j}F_jF_{k+1} & \sum_{j=1}^{n+1-k} F_{n-k-j}F_jF_k \end{bmatrix} \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{k=0}^{n} H_{2,n-k}RP^k &= \begin{bmatrix} \sum_{k=0}^{n}\sum_{j=1}^{n+1-k} F_{n+1-k-j}F_jF_{k+1} & \sum_{k=0}^{n}\sum_{j=1}^{n+1-k} F_{n+1-k-j}F_jF_k \\ \sum_{k=0}^{n}\sum_{j=1}^{n+1-k} F_{n-k-j}F_jF_{k+1} & \sum_{k=0}^{n}\sum_{j=1}^{n+1-k} F_{n-k-j}F_jF_k \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=1}^{n+1}\sum_{j=1}^{n+2-k} F_{n+2-k-j}F_jF_k & \sum_{k=1}^{n}\sum_{j=1}^{n+1-k} F_{n+1-k-j}F_jF_k \\ \sum_{k=1}^{n+1}\sum_{j=1}^{n+2-k} F_{n+1-k-j}F_jF_k & \sum_{k=1}^{n}\sum_{j=1}^{n+1-k} F_{n-k-j}F_jF_k \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i,j,k=n+2} F_iF_jF_k & \sum_{i,j,k=n+1} F_iF_jF_k \\ \sum_{i,j,k=n+1} F_iF_jF_k & \sum_{i,j,k=n} F_iF_jF_k \end{bmatrix} \end{aligned}$$

We then conclude the result. □

The main theorem can be done by follow these proofs recursively.

Theorem 2.1.4.

$$h_{m,n} = \sum_{\substack{a_i>0 \\ \sum a_i=n}} \prod_{i=1}^{m} F_{a_i}.$$

2.1.3 Binomial Sum of Fibonacci and Lucas Numbers

The following results are Lemma 3 in Demirturk and Kesking, [4].

Let R_b be a 2-by-2 matrix in the form $R_b := \begin{bmatrix} 1 & b \\ b & 1-b \end{bmatrix}$.

Proposition 2.1.5.

$$F_{mn+k} = \sum_j \binom{n}{j} F_m^j F_{m-1}^{n-j} F_{j+k},$$

and

$$L_{mn+k} = \sum_j \binom{n}{j} F_m^j F_{m-1}^{n-j} L_{j+k}.$$

Proof.

$$P^{mn+k}R_b = (P^m)^n P^k R_b = (F_m P + F_{m-1}I)^n P^k R_b = \left(\sum_j \binom{n}{j} F_m^j F_{m-1}^{n-j} P^j \right) P^k R_b$$

$$= \sum_j \binom{n}{j} F_m^j F_{m-1}^{n-j} P^{j+k} R_b.$$

Then set $b = 0$ for the first equation and $b = 2$ for the second equation. □

The followings are from the book, Proofs that really count by Benjamin and Quinn, page 144, [1].

Here we let $R := \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

1. (V69)

   $$5^n F_{2n-1} = \sum_{j=0}^{2n} \binom{2n}{j} F_{2j-1}.$$

   Consider $R^{2n}P^{2n-1} = P^{-1}(PR)^{2n} = P^{-1}(P^2+I)^{2n} = P^{-1}\sum_j \binom{2n}{j} P^{2j}$.

2. (V73)

   $$5^{n-1} L_{2n} = \sum_{j=0}^{2n} \binom{2n}{j} F_j^2, \quad n \geq 1.$$

   (I can't proof using matrix method.)

2.2 Gaussian Fibonacci Numbers

Let a 2-by-2 matrix R be in the form:

$$R := \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix}.$$

We define Fibonacci numbers, GF_n, by the following matrix:

$$\begin{bmatrix} GF_{n+1} & GF_n \\ GF_n & GF_{n-1} \end{bmatrix} := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix}.$$

Note:

1. The matrix is well defined.
2. It is obvious that $GF_{n+1} = GF_n + GF_{n-1}$ for $n \geq 1$. Moreover P and R commute.
3. $GF_n = F_n + iF_{n-1}$ by comparing the bottom left entry of $P^n R$.
4. $PR^2 = \begin{bmatrix} 1+2i & 0 \\ 0 & 1+2i \end{bmatrix}$.

2.2.1 Identities

The matrix form gives rise to the more compact form of "many" known identities. Most of the identities are from the paper by Jordan, [10].

1. $GF_1 = 1$ and $GF_0 = i$.
2. $GF_n = F_n + iF_{n-1}$.
3. $\sum_{j=0}^{n} GF_j = GF_{n+2} - 1$.
   From $1 + P + P^2 + ... + P^n = \dfrac{P^{n+1} - 1}{P - 1} = P^{n+2} - P$.
4. $GF_{n+1}GF_{n-1} - GF_n^2 = (-1)^n(2-i)$.
   Take determinant both sides of the definition.
5. $GF_n(GF_{n+1}+GF_{n-1}) = (1+2i)F_{2n-1}$. $\left(\text{Or } GF_{n+1}^2 - GF_{n-1}^2 = (1+2i)F_{2n-1}.\right)$

   Consider the top right entry of the following:

$$\begin{bmatrix} GF_{n+1} & GF_n \\ GF_n & GF_{n-1} \end{bmatrix} \cdot \begin{bmatrix} GF_{n+1} & GF_n \\ GF_n & GF_{n-1} \end{bmatrix} = (P^n R)^2 = P^{2n-1}(PR^2).$$

6. $GF_{n+1}GF_{m+1} + GF_nGF_m = (1+2i)F_{n+m}$.

   Consider the top right entry of the following:

$$\begin{bmatrix} GF_{n+1} & GF_n \\ GF_n & GF_{n-1} \end{bmatrix} \cdot \begin{bmatrix} GF_{m+2} & GF_{m+1} \\ GF_{m+1} & GF_m \end{bmatrix} = (P^nR)(P^{m+1}R) = P^{n+m}(PR^2).$$

   Special case $m = n$: $GF_{n+1}^2 + GF_n^2 = (1+2i)F_{2n}$.

7. $\sum_{j=1}^n GF_j^2 = (1+2i)F_n^2 + (-1)^n i - i$.
   Case when n is even:

$$\begin{aligned} \sum_{j=1}^n GF_j^2 &= \sum_{j=1}^n (F_j + iF_{j-1})^2 && \text{by identity (2)} \\ &= \sum_{j=1}^n (F_j^2 - F_{j-1}^2 + 2iF_jF_{j-1}) \\ &= F_n^2 + 2iF_n^2 && \text{by identity (5) of previous section} \\ &= (1+2i)F_n^2. \end{aligned}$$

   Case when n is odd:

$$\begin{aligned} \sum_{j=1}^n GF_j^2 &= \sum_{j=1}^n (F_j + iF_{j-1})^2 && \text{by identity (2)} \\ &= \sum_{j=1}^n (F_j^2 - F_{j-1}^2 + 2iF_jF_{j-1}) \\ &= F_n^2 + 2i(F_n^2 - 1) && \text{by identity (6) of previous section} \\ &= (1+2i)F_n^2 - 2i. \end{aligned}$$

8. $GF_{-n} = (-1)^n iGF_{n+1}$.
   Consider $P^{-n}R = (P^n)^{-1}R = (R^{-1}P^{-1}P^{n+1}R)^{-1}R$
   $= (P^{n+1}R)^{-1}PR^2 = (1+2i)(P^{n+1}R)^{-1}$.

9. $\sum_{j=1}^n GF_{2j-1} = GF_{2n} - i$.
   Consider the geometric sum $PR + P^3R + \cdots + P^{2n-1}R$.

10. $\sum_{j=1}^n GF_{2j} = GF_{2n+1} - 1$.
    Consider the geometric sum $P^2R + P^4R + \cdots + P^{2n}R$.

11. $\sum_{j=1}^{2n} (-1)^j GF_j = GF_{2n-1} - 1 + i$.
    Consider the geometric sum $-PR + P^2R + \cdots + P^{2n}R$.

12. $\sum_{j=1}^n (-1)^j GF_j = (-1)^n GF_{n-1} - 1 + i$.
    Consider the geometric sum $-PR + P^2R + \cdots + P^nR$.

2.3 Fibonacci Quaternions

Let a 2-by-2 matrix R be in the form:

$$R := \begin{bmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{bmatrix}$$

and let

$$P := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

We define Q_n and L_n as following:

Definition (Fibonacci Quaternions)**.**

$$\begin{bmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1+i+2j+3k & i+j+2k \\ i+j+2k & 1+j+k \end{bmatrix}.$$

Definition (Lucas numbers)**.**

$$\begin{bmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Again need to show that Q_n and L_n is well defined. (This can be done at the beginning once and for all.)

We will show some identities from the paper by Iyer, Some results on Fibonacci Quaternions, [8].

2.3.1 Identities from Iyer

1. $Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$.
   $L_n = 2F_{n+1} - F_n$.
   $F_n = \frac{1}{15}(Q_{n+1}(-4-3I+J-2K) + Q_n(7+4I-3J+K))$
   $F_n = \frac{1}{5}(2L_{n+1} - L_n)$.
   All these can be obtained from the definitions.

2. Identity (17)

   $Q_n - iQ_{n+1} - jQ_{n+2} - kQ_{n+3} = 3L_{n+3}$.
   Note that $I + iP + jP^2 + kP^3 = R$ and $I - iP - jP^2 - kP^3 = R^*$ (R with conjugate entries) and $R \cdot R^* = 3 \begin{bmatrix} L_4 & L_3 \\ L_3 & L_2 \end{bmatrix}$.

3. Identity (18)

   $Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n (2Q_1 - 3k)$.
   which can be obtained by taking the determinant on both sides of the definition. Also note that $Q_{-1}Q_1 - Q_0^2 = 2Q_1 - 3k$.

4. Identity (19)

$Q_n^2 + Q_{n-1}^2 = 2Q_{2n-1} - 3L_{2n+2}$.

Multiply $P^{-1}R$ to $R + R^* = 2I$ to get
$(P^{-1}R)R = 2P^{-1}R - P^{-1}R \cdot R^*$ which is

$$\begin{bmatrix} Q_0 & Q_{-1} \\ Q_{-1} & Q_{-2} \end{bmatrix} \cdot \begin{bmatrix} Q_1 & Q_0 \\ Q_0 & Q_{-1} \end{bmatrix} = 2 \begin{bmatrix} Q_0 & Q_{-1} \\ Q_{-1} & Q_{-2} \end{bmatrix} - 3 \begin{bmatrix} L_3 & L_2 \\ L_2 & L_1 \end{bmatrix}$$

then multiply $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{2n}$ on both sides.

5. Identity (20)

$Q_{n+1}Q_n + Q_nQ_{n-1} = Q_{n+1}^2 - Q_{n-1}^2 = (2Q_{2n} - 3L_{2n+3}) + 2(-1)^{n+1}(Q_0 - 3k)$.

Multiply R to $R + R^* = 2I$ to get $R^2 = 2R - R \cdot R^*$ which is

$$\begin{bmatrix} Q_1 & Q_0 \\ Q_0 & Q_{-1} \end{bmatrix}^2 = 2 \begin{bmatrix} Q_1 & Q_0 \\ Q_0 & Q_{-1} \end{bmatrix} - 3 \begin{bmatrix} L_4 & L_3 \\ L_3 & L_2 \end{bmatrix}$$

then multiply $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{2n}$ on both sides.

6. Identity (21)

$Q_nQ_{n+1} + Q_{n-1}Q_{n-2} = (6F_nQ_{n-1} - 9F_{2n+2}) + 2(-1)^{n+1}(Q_{-1} - 3k)$.

Note: $F_{n+1}F_n + F_{n-1}F_{n-2} = F_{2n} - F_{n-1}^2$. (This one I can prove)

7. Identity (22)

$Q_{n-1}Q_{n+3} - Q_{n+1}^2 = (-1)^n\,(2 + 4i + 6j + k)$.

$$\begin{bmatrix} Q_{n+3} & Q_n \\ Q_{n+2} & Q_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} Q_3 & Q_0 \\ Q_2 & Q_{-1} \end{bmatrix}$$

Take determinant on both sides to get:
$Q_{n+3}Q_{n-1} - Q_{n+2}Q_n = (-1)^n(Q_3Q_{-1} - Q_2Q_0)$.

Also
$$\begin{bmatrix} Q_{n+2} & Q_{n+1} \\ Q_{n+1} & Q_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} Q_2 & Q_1 \\ Q_1 & Q_0 \end{bmatrix}$$

Take determinant on both sides to get:
$Q_{n+2}Q_n - Q_{n+1}^2 = (-1)^n(Q_2Q_0 - Q_1^2)$.

Add the two equations to get:
$Q_{n+3}Q_{n-1} - Q_{n+1}^2 = (-1)^n(Q_3Q_{-1} - Q_1^2)$.

8. Identity (23)

$Q_{n+2}Q_{n-2} - Q_{n+1}Q_{n-1} = 2(-1)^n(-2 + I - J - 6K)$.
Consider

$$\begin{bmatrix} Q_{n+2} & Q_{n-1} \\ Q_{n+1} & Q_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} Q_2 & Q_{-1} \\ Q_1 & Q_{-2} \end{bmatrix}$$

Then take determinant on both sides to get:
$Q_{n+2}Q_{n-2} - Q_{n+1}Q_{n-1} = (-1)^n(Q_2Q_{-2} - Q_1Q_{-1})$.

9. Identity (24)

$Q_nQ_{n+1} + Q_{n-2}Q_{n-3} = 4Q_{2n-2} - 6L_{2n+1} + 2(-1)^{n+1}(i + j - k)$.

10. Identity (25)

$Q_{n+1}^2 + Q_{n-1}^2 = 6F_{n+1}Q_{n-1} - 9F_{2n+3} + 2(-1)^{n+1}(1 - i - k)$.

11. Identity (26)

$Q_{n+r} + (-1)^rQ_{n-r} = L_rQ_n$.
We want to show

$$\begin{bmatrix} Q_{n+1+r} & Q_{n+r} \\ Q_{n+r} & Q_{n-1+r} \end{bmatrix} + \begin{bmatrix} Q_{n+1-r} & Q_{n-r} \\ Q_{n-r} & Q_{n-1-r} \end{bmatrix} = L_r \begin{bmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{bmatrix}$$

when r is even. And

$$\begin{bmatrix} Q_{n+1+r} & Q_{n+r} \\ Q_{n+r} & Q_{n-1+r} \end{bmatrix} - \begin{bmatrix} Q_{n+1-r} & Q_{n-r} \\ Q_{n-r} & Q_{n-1-r} \end{bmatrix} = L_r \begin{bmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{bmatrix}$$

when r is odd.

These can be shown by
Case 1: r is even. Let $r = 2k$.
$P^{n+2k}R + P^{n-2k}R = (P^{2k} + P^{-2k}) \cdot P^nR = L_{2k}P^nR$.

Case 2: r is odd. Let $r = 2k + 1$.
$P^{n+2k+1}R - P^{n-2k-1}R = (P^{2k+1} - P^{-2k-1}) \cdot P^nR = L_{2k+1}P^nR$.

Note that $P^{2k} + P^{-2k} = L_{2k}$ and $P^{2k+1} - P^{-2k-1} = L_{2k+1}$ can be easily shown using an induction on k.
i.e. $P^{2k} + P^{-2k} = P^{2k-1} - P^{-2k+1} + P^{2k-2} + P^{-2k+2} = L_{2k-1} + L_{2k-2} = L_{2k}$.

2.3.2 Identities from Halici, [6]

1. Corollary 2(a)

   $$\sum_{i=0}^{n} Q_i = Q_{n+2} - Q_1.$$

   Same as Jor(1).

2. Corollary 2(b)

   $$\sum_{i=0}^{n} Q_{2i} = Q_{2n+1} - Q_{-1}.$$

   $(1 + P^2 + P^4 + \cdots + P^{2n})R = \dfrac{(P^{2n+2} - 1)R}{P} = P^{2n+1}R - P^{-1}R.$

3. Corollary 2(c)

   $$\sum_{i=0}^{n-1} Q_{2i+1} = Q_{2n} - Q_0.$$

   Same proof.

4. Theorem 3.5, equation (3.13)

   $\displaystyle\sum_{i=0}^{n} \binom{n}{i} Q_i = Q_{2n}.$

   $\sum \binom{n}{i} P^i R = (P^1 + P^0)^n R = P^{2n} R.$

5. Theorem 3.5, equation (3.14)

   $\displaystyle\sum_{i=0}^{n} (-1)^i \binom{n}{i} Q_i = (-1)^n Q_{-n}.$

   Similar proof.

2.4 $P^n R$ In General

We define S_n as following:

Definition.

$$\begin{bmatrix} S_{n+1} & S_n \\ S_n & S_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} a & b \\ b & a-b \end{bmatrix}.$$

It is possible to work with other *hyper complex numbers* other than Quaternions. However, for example, Cayley numbers are non-associative. With some extra care, we should be able to make it works.

2.5 Conclusion

This section serves like a warm-up problems. We use it to learn about the standard technique of matrix-form. The results may not very interesting and have been done before. Nonetheless, we mention it for completeness.

3 Two Matrix Multiply, $P^n R^m$

In this section we let a 2-by-2 matrix R be in the form:

$$R := \begin{bmatrix} a & b \\ b & a-b \end{bmatrix}$$

and consider the identities from matrices of the form $P^n R^m$.

3.1 Gaussian Fibonacci Numbers

Motivated by paper by Berzsenyi, Gaussian Fibonacci numbers, [2]. We define $F_{n,m}$ as following:

Definition.

$$\begin{bmatrix} F_{n+1,m} & F_{n,m} \\ F_{n,m} & F_{n-1,m} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix}^m.$$

We call $F_{n,m}$ as Gaussian Fibonacci numbers. Here we give a simpler proof than in the original paper. Some new identities have been discovered as well.

Corollary 3.1.1.

$$F_{n,m} = F_{n-1,m} + F_{n-2,m},$$

$$F_{n,m} = iF_{n+1,m-1} + (1-i)F_{n,m-1},$$

$$F_{n,m} = F_{n,m-1} + iF_{n-1,m-1},$$

$$F_{n,m} = F_{n+1,0}F_{0,m} + F_{n,0}F_{-1,m}.$$

for all $n \in \mathbb{Z}, m \in \mathbb{N}$ with the initial conditions: $F_{0,0} = 0, F_{1,0} = 1$.

Proof. From matrix multiplication and compare the entry on both sides. □

Corollary 3.1.2 (Generalize of the first Corollary)**.**

$$F_{n+s,m+t} = F_{n+1,m}F_{s,t} + F_{n,m}F_{s-1,t}.$$

Proof. Notice the following matrix multiplication then compare the entry on both sides.

$$\begin{bmatrix} F_{n+1,m} & F_{n,m} \\ F_{n,m} & F_{n-1,m} \end{bmatrix} \cdot \begin{bmatrix} F_{s+1,t} & F_{s,t} \\ F_{s,t} & F_{s-1,t} \end{bmatrix} = \begin{bmatrix} F_{n+s+1,m+t} & F_{n+s,m+t} \\ F_{n+s,m+t} & F_{n+s-1,m+t} \end{bmatrix}.$$

□

Some identities from the original paper which now can be done easily from the property of matrix multiplication.

1. $iF_{n+1,m} - F_{n,m+1} + (1-i)F_{n,m} = 0$, (monodiffric)
   From the definition

   $$\begin{bmatrix} F_{n+1,m+1} & F_{n,m+1} \\ F_{n,m+1} & F_{n-1,m+1} \end{bmatrix} = \begin{bmatrix} F_{n+1,m} & F_{n,m} \\ F_{n,m} & F_{n-1,m} \end{bmatrix} \cdot \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix}.$$

   Then compare the 12 entries.

2. $F_{n,m} = \sum_{k=0}^{m} \binom{m}{k} i^k F_{n-k,0}, \quad m \geq 0$

   Consider

   $$\begin{aligned} P^n R^m &= P^{n-m}(PR)^m \\ &= P^{n-m}(P+i)^m \\ &= P^{n-m} \sum_{k=0}^{m} \binom{m}{k} i^k P^{m-k} \\ &= \sum_{k=0}^{m} \binom{m}{k} i^k P^{n-k}. \end{aligned}$$

   This implies the statement.

3. $F_{m,2m} = 0$,

4. $F_{m+1,2m} = (1+2i)^m$,

   For (3) and (4), since $PR^2 = (1+2i)I$, we have

   $$P^m R^{2m} = (1+2i)^m I.$$

   Then (3) can be obtained by comparing the entry (1,2) on both sides. And (4) can be obtained by comparing the entry (1,1) on both sides.

5. $F_{n,2m} = F_{m+1,2m} F_{n-m,0} = (1+2i)^m F_{n-m,0}$,

   Consider

   $$\begin{aligned} P^n R^{2m} &= P^{n-m}(P^m R^{2m}) \\ &= (1+2i)^m P^{n-m}. \end{aligned}$$

   Then compare entry (1,2) on both sides.

6. $F_{n,2m+1} = (1+2i)^m \left(F_{n-m,0} + iF_{n-m-1,0}\right).$

Consider

$$\begin{aligned} P^n R^{2m+1} &= P^{n-m} R(P^m R^{2m}) \\ &= (1+2i)^m P^{n-m} R. \end{aligned}$$

Then compare entry (2,1) on both sides.

Corollary 3.1.3 (Generalize Cassini's identity)**.**

$$F_{n+1,m}F_{n-1,m} - F_{n,m}^2 = (-1)^n(2-i)^m.$$

Proof. By taking determinant on both sides of our definition. □

Vajda's identity generalize Cassini's identity.

Theorem 3.1.4 (Generalize Vajda's identity)**.**

$$F_{n+i,m+k}F_{n+j,m+l} - F_{n,m}F_{n+i+j,m+k+l} = (-1)^n(2-i)^m F_{i,k}F_{j,l}.$$

Proof. Take the determinant on both sides of the following equation:

$$\begin{bmatrix} F_{n+i,m+k} & F_{n,m} \\ F_{n+i+j,m+k+l} & F_{n+j,m+l} \end{bmatrix} = \begin{bmatrix} F_{n+1,0} & F_{n,0} \\ F_{n+j+1,l} & F_{n+j,l} \end{bmatrix} \cdot \begin{bmatrix} F_{i,m+k} & F_{0,m} \\ F_{i-1,m+k} & F_{-1,m} \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ F_{j+1,l} & F_{j,l} \end{bmatrix} \cdot \begin{bmatrix} F_{n+1,0} & F_{n,0} \\ F_{n,0} & F_{n-1,0} \end{bmatrix} \right) \cdot \left(\begin{bmatrix} F_{1,m} & F_{0,m} \\ F_{0,m} & F_{-1,m} \end{bmatrix} \cdot \begin{bmatrix} F_{i,k} & 0 \\ F_{i-1,k} & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 1 & 0 \\ F_{j+1,l} & F_{j,l} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \right) \cdot \left(\begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix}^m \cdot \begin{bmatrix} F_{i,k} & 0 \\ F_{i-1,k} & 1 \end{bmatrix} \right).$$

□

3.2 Generalize Lucas Numbers: $a = 1$ and $b = 2$

Definition.

$$\begin{bmatrix} S_{n+1,m} & S_{n,m} \\ S_{n,m} & S_{n-1,m} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^m.$$

Note:

1. $S_{n,0} = F_n$ and $S_{n,1} = L_n$.
2. $S_{1,m}$ = Inverse Binomial Mean transform of the Fibonacci sequence, $A074872$.
3. $P^2 = P + 1, R^2 = 5I$

4. $P^n R = L_n P + L_{n-1} I$

Theorem 3.2.1. $S_{n,m} = 2S_{n+1,m-1} - S_{n,m-1} = S_{n,m-1} + 2S_{n-1,m-1}$

Proof.

$$\begin{bmatrix} S_{n+1,m} & S_{n,m} \\ S_{n,m} & S_{n-1,m} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} S_{n+1,m-1} & S_{n,m-1} \\ S_{n,m-1} & S_{n-1,m-1} \end{bmatrix}.$$

□

Corollary 3.2.2.

$$L_n = F_n + 2F_{n-1}.$$

Proof. Set $m = 1$ in the theorem and compare the top right entries. □

Proposition 3.2.3. $S_{n,2m} = 5^m F_n$ *and* $S_{n,2m+1} = 5^m L_n$.

Proof. From $R^2 = 5I$. □

Corollary 3.2.4. $S_{0,2m} = 0$ *and* $S_{0,2m+1} = 2 \cdot 5^m$.

The following theorem and corollary are also a special case of theorem in the next section.

Theorem 3.2.5.

$$S_{n,m} = \sum_j \binom{m}{j} 2^j F_{n-j}.$$

Proof.

$$P^n R^m = P^{n-m}(PR)^m = P^{n-m}(P+2)^m = P^{n-m} \sum_j \binom{m}{j} 2^j P^{m-j} = \sum_j \binom{m}{j} 2^j P^{n-j}.$$

□

Corollary 3.2.6.

$$5^m F_n = \sum_j \binom{2m}{j} 2^j F_{n-j},$$

$$5^m L_n = \sum_j \binom{2m+1}{j} 2^j F_{n-j}.$$

3.3 Multiplication of the Two matrices

This is the generalization of [10], [8] and [2]. While [10], [8] are cases when $m = 1$ and reduces to case of one-matrix. [2] generalizes [10]. Here we try to generalize [8].

For a 2-by-2 matrix R to be commute with P, R must be in the form:

$$R := \begin{bmatrix} a & b \\ b & a-b \end{bmatrix}$$

We define $T_{n,m}$ as following:

Definition.

$$\begin{bmatrix} T_{n+1,m} & T_{n,m} \\ T_{n,m} & T_{n-1,m} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} a & b \\ b & a-b \end{bmatrix}^m.$$

for some constant a and b.

Corollary 3.3.1.

$$T_{n,m+1} = bT_{n+1,m} + (a-b)T_{n,m}.$$

Proof. This follows by comparing the lower left entries of

$$\begin{bmatrix} T_{n+1,m+1} & T_{n,m+1} \\ T_{n,m+1} & T_{n-1,m+1} \end{bmatrix} = \begin{bmatrix} a & b \\ b & a-b \end{bmatrix} \cdot \begin{bmatrix} T_{n+1,m} & T_{n,m} \\ T_{n,m} & T_{n-1,m} \end{bmatrix}$$

□

3.3.1 Some Identities!

Proposition 3.3.2 (Entries as a sum of two multiplications)**.**

$$T_{n+s,m+t} = T_{n+1,m}T_{s,t} + T_{n,m}T_{s-1,t} = T_{n,m}T_{s+1,t} + T_{n-1,m}T_{s,t}.$$

Proof. Notice the following matrix multiplication then compare the entries on both sides.

$$\begin{aligned} \begin{bmatrix} T_{n+s+1,m+t} & T_{n+s,m+t} \\ T_{n+s,m+t} & T_{n+s-1,m+t} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n+s} \cdot \begin{bmatrix} a & b \\ b & a-b \end{bmatrix}^{m+t} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n} \cdot \begin{bmatrix} a & b \\ b & a-b \end{bmatrix}^{m} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{s} \cdot \begin{bmatrix} a & b \\ b & a-b \end{bmatrix}^{t} \\ &= \begin{bmatrix} T_{n+1,m} & T_{n,m} \\ T_{n,m} & T_{n-1,m} \end{bmatrix} \cdot \begin{bmatrix} T_{s+1,t} & T_{s,t} \\ T_{s,t} & T_{s-1,t} \end{bmatrix}. \end{aligned}$$

□

Proposition 3.3.3 (Generalize Cassini's identity)**.**

$$T_{n+1,m}T_{n-1,m} - T_{n,m}^2 = (-1)^n(a^2 - ab - b^2)^m.$$

Proof. By taking the determinants on both sides of our definition. □

Theorem 3.3.4 (Generalize Vajda's identity)**.**

$$T_{n+i,m+k}T_{n+j,m+l} - T_{n,m}T_{n+i+j,m+k+l} = (-1)^n(a^2 - ab - b^2)^m T_{i,k}T_{j,l}.$$

Proof. Take the determinants on both sides of the following equation:

$$\begin{bmatrix} T_{n+i,m+k} & T_{n,m} \\ T_{n+i+j,m+k+l} & T_{n+j,m+l} \end{bmatrix} = \begin{bmatrix} T_{n+1,0} & T_{n,0} \\ T_{n+j+1,l} & T_{n+j,l} \end{bmatrix} \cdot \begin{bmatrix} T_{i,m+k} & T_{0,m} \\ T_{i-1,m+k} & T_{-1,m} \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ T_{j+1,l} & T_{j,l} \end{bmatrix} \cdot \begin{bmatrix} T_{n+1,0} & T_{n,0} \\ T_{n,0} & T_{n-1,0} \end{bmatrix} \right) \cdot \left(\begin{bmatrix} T_{1,m} & T_{0,m} \\ T_{0,m} & T_{-1,m} \end{bmatrix} \cdot \begin{bmatrix} T_{i,k} & 0 \\ T_{i-1,k} & 1 \end{bmatrix} \right).$$

$$= \left(\begin{bmatrix} 1 & 0 \\ T_{j+1,l} & T_{j,l} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \right) \cdot \left(\begin{bmatrix} a & b \\ b & a-b \end{bmatrix}^m \cdot \begin{bmatrix} T_{i,k} & 0 \\ T_{i-1,k} & 1 \end{bmatrix} \right).$$

□

3.4 Fibonacci as Sum of Binomial

It is possible to find a complex number b and integers A, B so that $P^A R^B$ is a diagonal matrix cI. I list their values in the table below:

A	B	b	c	Note
0	1	0	1	Identity
0	2	2	5	Lucas
1	2	$\pm i$	$1 \pm 2i$	Complex Fibo
2	1	-1	1	
2	2	$\frac{-1}{2}$	$\frac{5}{4}$	
2	4	$2 \pm \sqrt{5}$	$25(9 \pm 4\sqrt{5})$	
3	1	-2	-1	
3	2	$-1 \pm i$	$-1 \pm 2i$	
4	1	$-\frac{3}{2}$	$\frac{1}{2}$	
4	2	-3	5	
4	4	$\frac{-1 \pm \sqrt{5}i}{2}$	$-\frac{25}{4}$	
5	1	$-\frac{5}{3}$	$-\frac{1}{3}$	
5	2	$\frac{-3 \pm i}{2}$	$-1 \pm \frac{1}{2}i$	
...				

Theorem 3.4.1. *If there is an integers A, B and a complex number b satisfy the following:*

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^A \cdot \begin{bmatrix} 1 & b \\ b & 1-b \end{bmatrix}^B = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

then

$$c^m F_n = \sum_j \binom{Bm}{j} b^j F_{n+Am-j}, \text{ for any } n \in \mathbb{Z} \text{ and } m \in \mathbb{N}.$$

Proof. Consider the entries of $P^n \cdot R^{Bm}$
On one hand:

$$\begin{aligned} P^{n+Am} \cdot R^{Bm} &= P^n \cdot \left(P^A R^B\right)^m \\ &= c^m P^n. \end{aligned}$$

On the other hand:

$$\begin{aligned} P^{n+Am} \cdot R^{Bm} &= P^{n+Am-Bm} \cdot (PR)^{Bm} \\ &= P^{n+Am-Bm} \cdot (P+b)^{Bm} \\ &= P^{n+Am-Bm} \cdot \left(\sum_j \binom{Bm}{j} b^j P^{Bm-j} \right) \\ &= \sum_j \binom{Bm}{j} b^j P^{n+Am-j}. \end{aligned}$$

Then compare the right hand side of both equations to get the result. □

3.4.1 Sum Involving Binomial

We let $P := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $R_b := \begin{bmatrix} 1 & b \\ b & 1-b \end{bmatrix}$

We consider the identities arise from the matrix multiplication of the form

$$\begin{bmatrix} T_{n+1,m} & T_{n,m} \\ T_{n,m} & T_{n-1,m} \end{bmatrix} := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 & b \\ b & 1-b \end{bmatrix}^m.$$

The idea of two-matrix multiply is quite powerful. We start by using this method to show theorem 2 of Keskin and Demirturk, [4].

Proposition 3.4.2.

$$5^n F_{2mn+k} = \sum_j \binom{2n}{j} L_m^j L_{m-1}^{2n-j} F_{j+k},$$

$$5^n L_{2mn+k} = \sum_j \binom{2n}{j} L_m^j L_{m-1}^{2n-j} L_{j+k},$$

$$5^{n+1}F_{(2n+1)m+k} = \sum_j \binom{2n+1}{j} L_m^j L_{m-1}^{2n+1-j} L_{j+k},$$

and

$$5^n L_{(2n+1)m+k} = \sum_j \binom{2n+1}{j} L_m^j L_{m-1}^{2n+1-j} F_{j+k},$$

Proof. First note that $R_2^2 = 5I$.
For the first two equations,

$$\begin{aligned} R_2^{2n} P^{2mn+k} R_b &= (P^m R_2)^{2n} P^k R_b \\ &= (L_m P + L_{m-1} I)^{2n} P^k R_b \\ &= \sum_j \binom{2n}{j} L_m^j L_{m-1}^{2n-j} P^{j+k} R_b. \end{aligned}$$

Then set $b = 0$ for the first identity and $b = 2$ for the second identity .

The last two equations can be done similarly. □

The following theorem is the generalization of theorem 3.2.5, theorem 3.4.2 and theorem 5.3.2 .

Theorem 3.4.3. *Assume* $P^A R_b^B = cI$ *then*

$$c^{ns} T_{Bmn,0} = \sum_j \binom{Bn}{j} T_{m,s}^j T_{m-1,s}^{Bn-j} T_{Ans+j,0}.$$

Proof.

$$\begin{aligned} (P^A R_b^B)^{ns} P^{Bmn} &= (P^m R_b^s)^{Bn} P^{Ans} \\ &= (T_{m,s} P + T_{m-1,s} I)^{Bn} P^{Ans} \\ &= \sum_j \binom{Bn}{j} T_{m,s}^j T_{m-1,s}^{Bn-j} P^{Ans+j}. \end{aligned}$$

□

The following proposition is similar to the above theorem. We rewrite it in a more convenience form. Lemma 3 in Keskin and Demirturk, [4], is also a special case of this proposition.

Proposition 3.4.4. *Assume* $P^A R_b^B = cI$ *then*

$$c^n G_{(mB-A)n+k} = \sum_j \binom{Bn}{j} T_{m,1}^j T_{m-1,1}^{Bn-j} G_{j+k},$$

where $\begin{bmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot T.$

Proof. Consider

$$(R_b^B P^A)^n P^{(mB-A)n+k} T = (R_b P^m)^{Bn} P^k T = (T_{m,1} P + T_{m-1,1} I)^{Bn} P^k T$$
$$= \sum_j \binom{Bn}{j} T_{m,1}^j T_{m-1,1}^{Bn-j} P^{j+k} T.$$

□

Corollary 3.4.5 (Lemma 3 of Keskin and Demirturk, [4])**.**

Let $m, k \in \mathbb{Z}$ with $m \neq 1$ and $m \neq 0$. Then

$$F_{mn+k} = \sum_{j=0}^{n} \binom{n}{j} F_m^j F_{m-1}^{n-j} F_{j+k},$$

and

$$L_{mn+k} = \sum_{j=0}^{n} \binom{n}{j} F_m^j F_{m-1}^{n-j} L_{j+k},$$

for all $n \in \mathbb{N}$.

Proof. Notice that

$$P^2 R_{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore $P^m R_{-1} = P^{m-2}$ and $T_{m,1} = F_{m-2}$.
Now by the previous proposition, we have

$$G_{(m-2)n+k} = \sum_j \binom{n}{j} T_{m,1}^j T_{m-1,1}^{n-j} G_{j+k}$$

After substitute m by $m+2$, we have

$$G_{mn+k} = \sum_j \binom{n}{j} T_{m+2,1}^j T_{m+1,1}^{n-j} G_{j+k} = \sum_j \binom{n}{j} F_m^j F_{m+1}^{n-j} G_{j+k}.$$

Then let $T = I$ to obtain the first identity and let $T = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ to obtain the second identity. □

Another consequence of the proposition:

Corollary 3.4.6. *Assume $P^A R_b^B = cI$ for constants $A, B, b, c,$ then*

$$T_{mB+k,nB} = c^n F_{(mB-A)n+k}$$

Proof. Notice that $P^{mB+k} R_b^{nB} = (P^A R_b^B)^n P^{(mB-A)n+k}$.

□

Next are the identities among Fibonacci terms. (Although it might not be so spectacular.)

Proposition 3.4.7. *For any $n, m, s \in \mathbb{Z}$ and $b \in \mathbb{R}$,*

$$T_{n,m} = F_{n+1}G_m + F_nG_{m-1} = G_{m+1}F_n + G_mF_{n-1} = \sum_j \binom{m}{j} G_{s-1}^{m-j} G_s^j F_{n-sm+j}$$

where G_s is defined by $G_s = G_{s-1} + G_{s-2}$ where $G_0 = b, G_1 = 1$.

Proof.

$$\begin{aligned} P^nR^m &= (P^sR)^m P^{n-sm} \\ &= (G_sP + G_{s-1}I)^m P^{n-sm} \\ &= \left(\sum_j \binom{m}{j} G_{s-1}^{m-j} G_s^j P^j\right) P^{n-sm} \\ &= \sum_j \binom{m}{j} G_{s-1}^{m-j} G_s^j P^{n-sm+j}. \end{aligned}$$

□

Some special cases:

For $s = 0$, $\quad T_{n,m} = \sum_j \binom{m}{j} (1-b)^{m-j} b^j F_{n+j}$

For $s = 1$, $\quad T_{n,m} = \sum_j \binom{m}{j} b^{m-j} F_{n-m+j}$

For $s = 2$, $\quad T_{n,m} = \sum_j \binom{m}{j} (b+1)^j F_{n-2m+j}$

For $s = 3$, $\quad T_{n,m} = \sum_j \binom{m}{j} (b+1)^{m-j} (b+2)^j F_{n-3m+j}$.

4 Multiplication of Three (or more) Matrices

4.1 Multiplication of Three Matrices

Definition. We define $F_{n,m,p}$ as following

$$\begin{bmatrix} F_{n+1,m,p} & F_{n,m,p} \\ F_{n,m,p} & F_{n-1,m,p} \end{bmatrix} := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a & b \\ b & a-b \end{bmatrix}^m \begin{bmatrix} A & B \\ B & A-B \end{bmatrix}^p.$$

Proposition 4.1.1. $F_{n,m,p} = aF_{n,m-1,p} + bF_{n-1,m-1,p} = A \cdot F_{n,m,p-1} + B \cdot F_{n-1,m,p-1}$

Proof. Consider from the definition:

$$\begin{bmatrix} F_{n+1,m,p} & F_{n,m,p} \\ F_{n,m,p} & F_{n-1,m,p} \end{bmatrix} = \begin{bmatrix} F_{n+1,m-1,p} & F_{n,m-1,p} \\ F_{n,m-1,p} & F_{n-1,m-1,p} \end{bmatrix} \begin{bmatrix} a & b \\ b & a-b \end{bmatrix}$$
$$= \begin{bmatrix} F_{n+1,m,p-1} & F_{n,m,p-1} \\ F_{n,m,p-1} & F_{n-1,m,p-1} \end{bmatrix} \begin{bmatrix} A & B \\ A & A-B \end{bmatrix}.$$

□

Proposition 4.1.2.

$$F_{n,m,p} = F_{n+1,m,0}F_{0,0,p} + F_{n,m,0}F_{-1,0,p} = F_{n,m,0}F_{1,0,p} + F_{n-1,m,0}F_{0,0,p}$$
$$= F_{n+1,0,p}F_{0,m,0} + F_{n,0,p}F_{-1,m,0} = F_{n,0,p}F_{1,m,0} + F_{n-1,0,p}F_{0,m,0}.$$

Proof. Consider from the definition:

$$\begin{bmatrix} F_{n+1,m,p} & F_{n,m,p} \\ F_{n,m,p} & F_{n-1,m,p} \end{bmatrix} = \begin{bmatrix} F_{n+1,m,0} & F_{n,m,0} \\ F_{n,m,0} & F_{n-1,m,0} \end{bmatrix} \begin{bmatrix} F_{1,0,p} & F_{0,0,p} \\ F_{0,0,p} & F_{-1,0,p} \end{bmatrix}$$
$$= \begin{bmatrix} F_{n+1,0,p} & F_{n,0,p} \\ F_{n,0,p} & F_{n-1,0,p} \end{bmatrix} \begin{bmatrix} F_{1,m,0} & F_{0,m,0} \\ F_{0,m,0} & F_{-1,m,0} \end{bmatrix}.$$

□

Proposition 4.1.3 (Generalize Cassini's identity)**.**

$$F_{n+1,m,p}F_{n-1,m,p} - F_{n,m,p}^2 = (-1)^n(a^2 - ab - b^2)^m(A^2 - AB - B^2)^p.$$

Proof. By taking determinant on both sides of the definition. □

Theorem 4.1.4 (Generalize Vajda's identity)**.**

$$F_{n+i_1,m+i_2,p+i_3}F_{n+j_1,m+j_2,p+j_3} - F_{n,m,p}F_{n+i_1+j_1,m+i_2+j_2,p+i_3+j_3}$$

$$= (-1)^n(a^2 - ab - b^2)^m(A^2 - AB - B^2)^p F_{i_1,i_2,i_3}F_{j_1,j_2,j_3}.$$

Proof.

$$\begin{bmatrix} F_{n+i_1,m+i_2,p+i_3} & F_{n,m,p} \\ F_{n+i_1+j_1,m+i_2+j_2,p+i_3+j_3} & F_{n+j_1,m+j_2,p+j_3} \end{bmatrix}$$
$$= \begin{bmatrix} F_{n+1,0,0} & F_{n,0,0} \\ F_{n+j_1+1,j_2,j_3} & F_{n+j_1,j_2,j_3} \end{bmatrix} \cdot \begin{bmatrix} F_{i_1,m+i_2,p+i_3} & F_{0,m,p} \\ F_{i_1-1,m+i_2,p+i_3} & F_{-1,m,p} \end{bmatrix}$$
$$= \left(\begin{bmatrix} 1 & 0 \\ F_{j_1+1,j_2,j_3} & F_{j_1,j_2,j_3} \end{bmatrix} \cdot \begin{bmatrix} F_{n+1,0,0} & F_{n,0,0} \\ F_{n,0,0} & F_{n-1,0,0} \end{bmatrix} \right) \cdot \left(\begin{bmatrix} F_{1,m,p} & F_{0,m,p} \\ F_{0,m,p} & F_{-1,m,p} \end{bmatrix} \cdot \begin{bmatrix} F_{i_1,i_2,i_3} & 0 \\ F_{i_1-1,i_2,i_3} & 1 \end{bmatrix} \right)$$
$$= \left(\begin{bmatrix} 1 & 0 \\ F_{j_1+1,j_2,j_3} & F_{j_1,j_2,j_3} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \right) \cdot \left(\begin{bmatrix} a & b \\ b & a-b \end{bmatrix}^m \cdot \begin{bmatrix} A & B \\ B & A-B \end{bmatrix}^p \cdot \begin{bmatrix} F_{i_1,i_2,i_3} & 0 \\ F_{i_1-1,i_2,i_3} & 1 \end{bmatrix} \right),$$

Then take determinant on both sides. □

4.2 Quaternions

Next, we generalize Berzsenyi results, [2], with Quaternions. This one is harder since the element in the matrix does not commute.

Definition. We define $Q_{n,m,p,q}$ as following

$$\begin{bmatrix} Q_{n+1,m,p,q} & Q_{n,m,p,q} \\ Q_{n,m,p,q} & Q_{n-1,m,p,q} \end{bmatrix} := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix}^m \begin{bmatrix} 1 & j \\ j & 1-j \end{bmatrix}^p \begin{bmatrix} 1 & k \\ k & 1-k \end{bmatrix}^q.$$

Corollary 4.2.1.

$$Q_{n,m,p,q} = Q_{n-1,m,p,q} + Q_{n-2,m,p,q},$$

$$Q_{n,m,p,q} = iQ_{n+1,m-1,p,q} + (1-i)Q_{n,m-1,p,q},$$

$$Q_{n,m,p,q} = Q_{n,m-1,p,q} + iQ_{n-1,m-1,p,q},$$

$$Q_{n,m,p,q} = F_{n+1}Q_{0,m,p,q} + F_nQ_{-1,m,p,q}.$$

$$Q_{n,m,p,q} = Q_{n+1,m,0,0}Q_{0,0,p,q} + Q_{n,m,0,0}Q_{-1,0,p,q}.$$

Proof. From matrix multiplication and compare the entry on both sides. □

Corollary 4.2.2 (Generalize of the first Corollary)**.**

$$Q_{n_1+n_2,m_1+m_2,p_1+p_2,q_1+q_2} = Q_{n_1+1,m_1,p_1,q_1}Q_{n_2,m_2,p_2,q_2} + Q_{n_1,m_1,p_1,q_1}Q_{n_2-1,m_2,p_2,q_2}.$$

Proof. Notice the following matrix multiplication then compare the entry on both sides.

$$\begin{bmatrix} Q_{n_1+1,m_1,p_1,q_1} & Q_{n_1,m_1,p_1,q_1} \\ Q_{n_1,m_1,p_1,q_1} & Q_{n_1-1,m_1,p_1,q_1} \end{bmatrix} \cdot \begin{bmatrix} Q_{n_2+1,m_2,p_2,q_2} & Q_{n_2,m_2,p_2,q_2} \\ Q_{n_2,m_2,p_2,q_2} & Q_{n_2-1,m_2,p_2,q_2} \end{bmatrix}$$

$$= \begin{bmatrix} Q_{n_1+n_2+1,m_1+m_2,p_1+p_2,q_1+q_2} & Q_{n_1+n_2,m_1+m_2,p_1+p_2,q_1+q_2} \\ Q_{n_1+n_2,m_1+m_2,p_1+p_2,q_1+q_2} & Q_{n_1+n_2-1,m_1+m_2,p_1+p_2,q_1+q_2} \end{bmatrix}.$$

□

The following identities can be shown in a straight forward way using the matrix.

Theorem 4.2.3 (Generalize the results of Berzsenyi, [2])**.**

1. $Q_{n,m+1,p,q} - Q_{n,m,p,q} = i(Q_{n+1,m,p,q} - Q_{n,m,p,q})$, *(monodiffric)*
   $Q_{n,m,p,q+1} - Q_{n,m,p,q} = (Q_{n+1,m,p,q} - Q_{n,m,p,q})k$,
   By (1) and (2), we get:
   $i[Q_{n,m,p,q+1} - Q_{n,m,p,q}] = [Q_{n,m+1,p,q} - Q_{n,m,p,q}]k.$
2. $Q_{n,m,p,q} = \sum_{s_1,s_2,s_3} \binom{m}{s_1}\binom{p}{s_2}\binom{q}{s_3} i^{s_1} j^{s_2} k^{s_3} F_{n-s_1-s_2-s_3}$,
3. $Q_{m,2m,0,0} = Q_{p,0,2p,0} = Q_{q,0,0,2q} = 0$,
4. $Q_{m\pm1,2m,0,0} = (1+2i)^m, Q_{p\pm1,0,2p,0} = (1+2j)^p, Q_{q\pm1,0,0,2q} = (1+2k)^q$.

5. $Q_{n,2m,p,q} = Q_{m+1,2m,0,0}Q_{n-m,0,p,q} = (1+2i)^m Q_{n-m,0,p,q}$,

6. $Q_{n,2m+1,p,q} = (1+2i)^m \left[Q_{n-m,0,p,q} + iQ_{n-m-1,0,p,q}\right]$.

Proof. Note that $N^2 = N + I$ and $N^2M^2 = (N + iI)^2 = (1+2i)N$ which implies $NM^2 = 1 + 2i$. Also $NM = N + i$.

2) $N^n M^m P^p Q^q$

$$\begin{aligned}
&= N^{n-m-p-q} \cdot [NM]^m \cdot [NP]^p \cdot [NQ]^q \\
&= N^{n-m-p-q} \cdot [N+i]^m \cdot [N+j]^p \cdot [N+k]^q \\
&= N^{n-m-p-q} \cdot \left(\sum_{s_1=0}^{m} \binom{m}{s_1} i^{s_1} N^{m-s_1}\right) \cdot \left(\sum_{s_2=0}^{p} \binom{p}{s_2} j^{s_2} N^{p-s_2}\right) \cdot \left(\sum_{s_3=0}^{q} \binom{q}{s_3} k^{s_3} N^{q-s_3}\right) \\
&= \sum_{s_1,s_2,s_3} \binom{m}{s_1}\binom{p}{s_2}\binom{q}{s_3} i^{s_1} j^{s_2} k^{s_3} N^{n-s_1-s_2-s_3}.
\end{aligned}$$

3) and 4) follows directly from $(NM^2)^m = (1+2i)^m$.
5) follows from $N^n M^{2m} P^p Q^q = (NM^2)^m \cdot N^{n-m} P^p Q^q$.

$$\begin{bmatrix} Q_{n+1,2m,p,q} & Q_{n,2m,p,q} \\ Q_{n,2m,p,q} & Q_{n-1,2m,p,q} \end{bmatrix} = \begin{bmatrix} Q_{m+1,2m,0,0} & Q_{m,2m,0,0} \\ Q_{m,2m,0,0} & Q_{m-1,2m,0,0} \end{bmatrix} \begin{bmatrix} Q_{n-m+1,0,p,q} & Q_{n-m,0,p,q} \\ Q_{n-m,0,p,q} & Q_{n-m-1,0,p,q} \end{bmatrix}.$$

6) follows from $N^n M^{2m+1} P^p Q^q = M \cdot (NM^2)^m \cdot (N^{n-m} P^p Q^q)$.

$$\begin{aligned}
&\begin{bmatrix} Q_{n+1,2m+1,p,q} & Q_{n,2m+1,p,q} \\ Q_{n,2m+1,p,q} & Q_{n-1,2m+1,p,q} \end{bmatrix} \\
&= \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix} \begin{bmatrix} Q_{m+1,2m,0,0} & Q_{m,2m,0,0} \\ Q_{m,2m,0,0} & Q_{m-1,2m,0,0} \end{bmatrix} \begin{bmatrix} Q_{n-m+1,0,p,q} & Q_{n-m,0,p,q} \\ Q_{n-m,0,p,q} & Q_{n-m-1,0,p,q} \end{bmatrix} \\
&= \begin{bmatrix} 1 & i \\ i & 1-i \end{bmatrix} \begin{bmatrix} Q_{m+1,2m,0,0}Q_{n-m+1,0,p,q} & Q_{m+1,2m,0,0}Q_{n-m,0,p,q} \\ Q_{m-1,2m,0,0}Q_{n-m,0,p,q} & Q_{m-1,2m,0,0}Q_{n-m-1,0,p,q} \end{bmatrix}.
\end{aligned}$$

and compare the entry on the bottom left entry.

□

Corollary 4.2.4 (Generalize Cassini's identity)**.**

$$Q_{n+1,m,0,0}Q_{n-1,m,0,0} - Q_{n,m,0,0}^2 = (-1)^n(2-i)^m.$$

Proof. By taking determinant on both sides of our definition. □

Remark 1. No more general version of Cassini's identity as the entry in the matrix does not commute.

5 Numbers related to 3-by-3 Matrix

5.1 Tribonacci Numbers

By letting $M := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, we also have $M^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$.

Then Tribonacci numbers,

$$t_n := t_{n-1} + t_{n-2} + t_{n-3} \text{ where } t_0 = t_1 = 0 \text{ and } t_2 = 1$$

can be defined using this matrix by

Let $\mathcal{A}_n := \begin{bmatrix} t_{n+2} & t_{n+1} + t_n & t_{n+1} \\ t_{n+1} & t_n + t_{n-1} & t_n \\ t_n & t_{n-1} + t_{n-2} & t_{n-1} \end{bmatrix}$.

Then $\mathcal{A}_n = M^n$ and we also have the relation $\mathcal{A}_{n+1} = M\mathcal{A}_n = \mathcal{A}_n M$.

5.1.1 Properties

1. $t_{-n} = \begin{vmatrix} t_{n+1} & t_{n+2} \\ t_n & t_{n+1} \end{vmatrix}$.

   From $M^{-n} = (M^n)^{-1}$:

$$t_{-n} = \frac{1}{|M|^n} \begin{vmatrix} t_{n+1} & t_n + t_{n-1} \\ t_n & t_{n-1} + t_{n-2} \end{vmatrix} = \begin{vmatrix} t_{n+1} & t_{n+1} + t_n + t_{n-1} \\ t_n & t_n + t_{n-1} + t_{n-2} \end{vmatrix} = \begin{vmatrix} t_{n+1} & t_{n+2} \\ t_n & t_{n+1} \end{vmatrix}.$$

2. $\begin{vmatrix} t_{n+1} & t_{n+2} & t_{n+3} \\ t_n & t_{n+1} & t_{n+2} \\ t_{n-1} & t_n & t_{n+1} \end{vmatrix} = 1.$

$$\begin{aligned} 1 = |M^n| &= \begin{vmatrix} t_{n+2} & t_{n+1} + t_n & t_{n+1} \\ t_{n+1} & t_n + t_{n-1} & t_n \\ t_n & t_{n-1} + t_{n-2} & t_{n-1} \end{vmatrix} = \begin{vmatrix} t_{n+2} & t_{n+2} + t_{n+1} + t_n & t_{n+1} \\ t_{n+1} & t_{n+1} + t_n + t_{n-1} & t_n \\ t_n & t_n + t_{n-1} + t_{n-2} & t_{n-1} \end{vmatrix} \\ &= \begin{vmatrix} t_{n+2} & t_{n+3} & t_{n+1} \\ t_{n+1} & t_{n+2} & t_n \\ t_n & t_{n+1} & t_{n-1} \end{vmatrix} = \begin{vmatrix} t_{n+1} & t_{n+2} & t_{n+3} \\ t_n & t_{n+1} & t_{n+2} \\ t_{n-1} & t_n & t_{n+1} \end{vmatrix}. \end{aligned}$$

3. $t_{n+m} = t_n t_{m+2} + (t_{n-1} + t_{n-2})t_{m+1} + t_{n-1}t_m$.

   Consider the (3,1) entry of M^{n+m},

$$M^{n+m} = \begin{bmatrix} t_{n+2} & t_{n+1} + t_n & t_{n+1} \\ t_{n+1} & t_n + t_{n-1} & t_n \\ t_n & t_{n-1} + t_{n-2} & t_{n-1} \end{bmatrix} \cdot \begin{bmatrix} t_{m+2} & t_{m+1} + t_m & t_{m+1} \\ t_{m+1} & t_m + t_{m-1} & t_m \\ t_m & t_{m-1} + t_{m-2} & t_{m-1} \end{bmatrix}.$$

5.2 Trucas Numbers

$$u_n := u_{n-1} + u_{n-2} + u_{n-3} \text{ where } u_0 = 3, u_1 = 1 \text{ and } u_2 = 3$$

can be defined using the matrix.

$$\text{Define } U_n := \begin{bmatrix} u_{n+2} & u_{n+1} + u_n & u_{n+1} \\ u_{n+1} & u_n + u_{n-1} & u_n \\ u_n & u_{n-1} + u_{n-2} & u_{n-1} \end{bmatrix}.$$

5.2.1 Properties

1. $U_n = M^n \cdot \begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{bmatrix}.$

2. $u_n = t_{n+1} + 2t_n + 3t_{n-1}.$

   Noticed that $\begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{bmatrix} = M + 2I + 3M^{-1}.$

3. $\begin{vmatrix} u_{n+1} & u_{n+2} & u_{n+3} \\ u_n & u_{n+1} & u_{n+2} \\ u_{n-1} & u_n & u_{n+1} \end{vmatrix} = 44 \quad \text{for all integer } n.$

4. $t_n = \frac{1}{22}(2u_n + u_{n-1} + 5u_{n-2}) = \frac{1}{22}(5u_{n+1} - 3u_n - 4u_{n-1}).$

   Notice that $\begin{bmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{bmatrix}^{-1} = \frac{1}{22}\begin{bmatrix} 2 & 1 & 5 \\ 5 & -3 & -4 \\ -4 & 9 & 1 \end{bmatrix}.$

   Therefore

$$\begin{aligned} \begin{bmatrix} t_{n+2} & t_{n+1} + t_n & t_{n+1} \\ t_{n+1} & t_n + t_{n-1} & t_n \\ t_n & t_{n-1} + t_{n-2} & t_{n-1} \end{bmatrix} &= \frac{1}{22}\begin{bmatrix} u_{n+2} & u_{n+1} + u_n & u_{n+1} \\ u_{n+1} & u_n + u_{n-1} & u_n \\ u_n & u_{n-1} + u_{n-2} & u_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 5 \\ 5 & -3 & -4 \\ -4 & 9 & 1 \end{bmatrix} \\ &= \frac{1}{22}\begin{bmatrix} 2 & 1 & 5 \\ 5 & -3 & -4 \\ -4 & 9 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{n+2} & u_{n+1} + u_n & u_{n+1} \\ u_{n+1} & u_n + u_{n-1} & u_n \\ u_n & u_{n-1} + u_{n-2} & u_{n-1} \end{bmatrix}. \end{aligned}$$

5.3 Multiplication of Two Matrices

We found the new relations by generalizing the matrix of the form $M^A R^B$.

Lemma 5.3.1. *Let the 3-by-3 matrix M be* $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ *and let the 3-by-3 matrix R be* $\begin{bmatrix} 1 & a+b & b \\ b & 1-b & a \\ a & b-a & 1-b-a \end{bmatrix}$.

Then $MR = RM = M + (a+b) \cdot I + b \cdot M^{-1}$.

Proof. This can be verified directly. □

Using the same notation, below is the table of $M^A \cdot R^B = cI$.

A	B	a	b	c	Note
0	1	0	0	1	Identity
3	1	0	-1	1	
4	1	-2	0	-1	
6	1	$\frac{1}{2}$	$\frac{-3}{2}$	$\frac{1}{2}$	
7	1	$-\frac{4}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	
8	1	-8	4	1	
9	1	$\frac{5}{4}$	-2	$\frac{1}{4}$	
10	1	$-\frac{7}{8}$	$-\frac{5}{8}$	$-\frac{1}{8}$	
1	3	-1	1	4	
2	3	E	$-\frac{1}{3}E$	$\frac{1}{16}(121 \pm \frac{143\sqrt{33}}{9})$	$E = \dfrac{3 \pm \sqrt{33}}{4}$
3	3	$-\frac{2}{3}E - \frac{1}{3}$	E	$121 \pm \frac{187\sqrt{33}}{9}$	$E = \dfrac{5 \pm \sqrt{33}}{2}$
4	3	1	-1	4	
5	3	-1	0	2	
6	3				
7	3	-1	-1	-4	
8	3				
10	3	-3	1	4	
11	3	1	-2	-2	
13	3	3	-3	4	
-2	3	1	1	4	Lucas
...					

The application of the table above is as below:

Theorem 5.3.2. *If there is an integers A, B and a complex number b satisfy the following:*

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^A \cdot \begin{bmatrix} 1 & a+b & b \\ b & 1-b & a \\ a & b-a & 1-b-a \end{bmatrix}^B = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}.$$

then

$$c^m t_n = \sum_{i,j} \binom{Bm}{i, j, Bm-i-j} (a+b)^i b^j t_{n+Am-i-2j}$$

, for any $n \in \mathbb{Z}$ *and* $m \in \mathbb{N}$.

Proof. Consider the entries of $M^{n+Am} \cdot R^{Bm}$
On one hand:

$$\begin{aligned} M^{n+Am} \cdot R^{Bm} &= M^n \cdot \left(M^A R^B\right)^m \\ &= c^m M^n. \end{aligned}$$

On the other hand:

$$\begin{aligned} M^{n+Am} \cdot R^{Bm} &= M^{n+Am-Bm} \cdot (MR)^{Bm} \\ &= M^{n+Am-Bm} \cdot (M + (a+b)I + bM^{-1})^{Bm} \\ &= M^{n+Am-Bm} \cdot \left(\sum_{i,j} \binom{Bm}{i, j, Bm-i-j} (a+b)^i b^j M^{Bm-i-2j} \right) \\ &= \sum_{i,j} \binom{Bm}{i, j, Bm-i-j} (a+b)^i b^j M^{n+Am-i-2j}. \end{aligned}$$

Then compare the right hand side of both equations to get the result. □

6 Another Form of Complex Fibonacci Numbers

We find another matrix representation of the results presented by Harman, [7]. We can define $G(n,m)$ as an entry in the matrix below:

$$\mathcal{A}(n,m) := \begin{bmatrix} G(n+1,m+1) & G(n,m+1) \\ G(n+1,m) & G(n,m) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^m \cdot \begin{bmatrix} 1+i & i \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n.$$

Remark 2. The matrix is well defined. (which can be shown by 1. and 2. below)

6.1 Identities

1. With $n = 0, m = 0$, we have that $G(0,0) = 0$, $G(1,0) = 1$, $G(0,1) = i$ and $G(1,1) = 1 + i$.

2. $G(n+1,m) = G(n,m) + G(n-1,m)$ and $G(n,m+1) = G(n,m) + G(n,m-1)$.

   Consider (2,1) entry:

$$\begin{bmatrix} G(n+1,m+1) & G(n,m+1) \\ G(n+1,m) & G(n,m) \end{bmatrix} = \begin{bmatrix} G(n,m+1) & G(n-1,m+1) \\ G(n,m) & G(n-1,m) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Consider (1,2) entry:

$$\begin{bmatrix} G(n+1,m+1) & G(n,m+1) \\ G(n+1,m) & G(n,m) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} G(n+1,m) & G(n,m) \\ G(n+1,m-1) & G(n,m-1) \end{bmatrix}.$$

3. $G(n+1,m+1) = G(n,m) + G(n,m-1) + G(n-1,m) + G(n-1,m-1)$.

   Follows from the previous.

4. $G(n,0) = F_n$ and $G(0,m) = iF_m$.

   This can be shown using $G(n+2,0) = G(n+1,0) + G(n,0)$ and $G(0,m+2) = G(0,m+1) + G(0,m)$ along the initial conditions.

5. $G(n,m) = F_m G(n,1) + F_{m-1} G(n,0) = F_n G(1,m) + F_{n-1} G(0,m)$.

   Consider the (2,2) entry in

$$\begin{aligned} \begin{bmatrix} G(n+1,m+1) & G(n,m+1) \\ G(n+1,m) & G(n,m) \end{bmatrix} &= \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix} \cdot \begin{bmatrix} G(n+1,1) & G(n,1) \\ G(n+1,0) & G(n,0) \end{bmatrix} \\ &= \begin{bmatrix} G(1,m+1) & G(0,m+1) \\ G(1,m) & G(0,m) \end{bmatrix} \cdot \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}. \end{aligned}$$

6. $G(n,m) = F_n F_{m+1} + iF_{n+1} F_m$.

   Consider the (2,2) entry in

$$\begin{aligned} \begin{bmatrix} G(n+1,m+1) & G(n,m+1) \\ G(n+1,m) & G(n,m) \end{bmatrix} &= \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix} \cdot \begin{bmatrix} 1+i & i \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} F_{m+2} + iF_{m+1} & iF_{m+1} \\ F_{m+1} + iF_m & iF_m \end{bmatrix} \cdot \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \end{aligned}$$

7. $G(n,m) + G(m,n) = (1+i)(F_m F_{n+1} + F_{m+1} F_n)$.

   The result follows immediately from the previous.

8. $G(n,m) + G(n-1,m-1) = (1+i) F_{m+n}$.

   Consider the (2,2) entry in

$$\begin{aligned} \mathcal{A}(n,m) + \mathcal{A}(n-1,m-1) &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^m \cdot \begin{bmatrix} 1+i & 1+i \\ 1+i & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \\ &= \begin{bmatrix} 1+i & 0 \\ 0 & 1+i \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{m+n+1}. \end{aligned}$$

9. $G(n,m) = (1+i)F_nF_m + G(m-1,n-1)$

   By combining the previous 2 results.

10. $G(n+2k,m+2k) = (1+i)\sum_{j=1}^{2k} F_{n+j}F_{m+j} + G(n,m)$

   Telescoping version of 9.

11. $G(n+2k+1,m+2k+1) = (1+i)\sum_{j=1}^{2k+1} F_{n+j}F_{m+j} + G(m,n)$

   Telescoping version of 9.

6.2 Constant matrix representation of the symmetric form

We also show that the matrix can generated by a constant matrix.

$$\begin{bmatrix} G(n,1) & G(n,0) \\ G(n,0) & G(n,-1) \end{bmatrix} = \begin{bmatrix} F_n & F_n \\ F_n & 0 \end{bmatrix} + \begin{bmatrix} iF_{n+1} & 0 \\ 0 & iF_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} - \begin{bmatrix} F_{n-1} & 0 \\ 0 & F_{n-1} \end{bmatrix} + \begin{bmatrix} iF_{n+1} & 0 \\ 0 & iF_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n - D(S^{n-1} - T^{n-1}) + \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \cdot D(S^{n+1} - T^{n+1})$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n - S^{n-1}D\left(I - \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \cdot S^2\right) + T^{n-1}D\left(I - \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \cdot T^2\right)$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}^n + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}^n.$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $a = \frac{-(5-\sqrt{5})+(5+\sqrt{5})i}{10}$ and $b = \frac{-(5+\sqrt{5})+(5-\sqrt{5})i}{10}$.

Second we write the matrix product for the general matrix.

$$\begin{bmatrix} G(n,m+1) & G(n,m) \\ G(n,m) & G(n,m-1) \end{bmatrix} = \begin{bmatrix} G(n,1) & G(n,0) \\ G(n,0) & G(n,-1) \end{bmatrix} \cdot \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n - \begin{bmatrix} F_{n-1} & 0 \\ 0 & F_{n-1} \end{bmatrix} + \begin{bmatrix} iF_{n+1} & 0 \\ 0 & iF_{n+1} \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^m.$$

6.3 Third Order Recurrence

In this section we define $H(n,m)$ as an entry in the matrix of the following:

$$\mathcal{B}(n,m) := \begin{bmatrix} H(n+2,m+2) & H(n+1,m+2) & H(n,m+2) \\ H(n+2,m+1) & H(n+1,m+1) & H(n,m+1) \\ H(n+2,m) & H(n+1,m) & H(n,m) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^m \cdot \begin{bmatrix} 2+2i & 2+i & i \\ 1+2i & 1+i & i \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^n.$$

Remark 3. The matrix is well defined. (which can be shown by 1. and 2. below)

6.3.1 Identities

1. With $n=0, m=0$, we have the following as initial conditions
$H(0,0)=0,\ \ H(1,0)=1,\ \ H(2,0)=1,\ \ H(0,1)=i,$
$H(1,1)=1+i,\ \ H(2,1)=1+2i,\ \ H(0,2)=i,\ \ H(1,2)=2+i$ and
$H(2,2)=2+2i.$

2. $H(n+2,m)=H(n+1,m)+H(n,m)+H(n-1,m)$ and
$H(n,m+2)=H(n,m+1)+H(n,m)+H(n,m-1).$

   Consider (1,1), (2,1) and (3,1) entries:

$$\begin{bmatrix} H(n+2,m+2) & H(n+1,m+2) & H(n,m+2) \\ H(n+2,m+1) & H(n+1,m+1) & H(n,m+1) \\ H(n+2,m) & H(n+1,m) & H(n,m) \end{bmatrix} = \begin{bmatrix} H(n+1,m+2) & H(n,m+2) & H(n-1,m+2) \\ H(n+1,m+1) & H(n,m+1) & H(n-1,m+1) \\ H(n+1,m) & H(n,m) & H(n-1,m) \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

   Consider (1,1), (1,2) and (1,3) entries:

$$\begin{bmatrix} H(n+2,m+2) & H(n+1,m+2) & H(n,m+2) \\ H(n+2,m+1) & H(n+1,m+1) & H(n,m+1) \\ H(n+2,m) & H(n+1,m) & H(n,m) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} H(n+2,m+1) & H(n+1,m+1) & H(n,m+1) \\ H(n+2,m) & H(n+1,m) & H(n,m) \\ H(n+2,m-1) & H(n+1,m-1) & H(n,m-1) \end{bmatrix}.$$

3.

$$\begin{aligned} H(n+1,m+1) =& H(n+1,m+1)+H(n+1,m)+H(n+1,m-1)+ \\ & H(n,m+1)+H(n,m)+H(n,m-1)+ \\ & H(n-1,m+1)+H(n-1,m)+H(n-1,m-1). \end{aligned}$$

Follows from the previous identity.

4. $H(n,0) = t_{n+1}$ and $H(0,m) = it_{m+1}$.

   This can be shown using $H(n+2,0) = H(n+1,0) + H(n,0) + H(n-1,0)$ and $H(0,m+2) = H(0,m+1) + H(0,m) + H(0,m-1)$ along the initial conditions.

5. $H(n,m) = t_{m+1}H(n,1) + (t_m + t_{m-1})H(n,0) = t_{n+1}H(1,m) + (t_n + t_{n-1})H(0,m)$.

   Consider the (3,3) entry in

$$\begin{bmatrix} H(n+2,m+2) & H(n+1,m+2) & H(n,m+2) \\ H(n+2,m+1) & H(n+1,m+1) & H(n,m+1) \\ H(n+2,m) & H(n+1,m) & H(n,m) \end{bmatrix}$$
$$= \begin{bmatrix} t_{m+3} & t_{m+2}+t_{m+1} & t_{m+2} \\ t_{m+2} & t_{m+1}+t_m & t_{m+1} \\ t_{m+1} & t_m+t_{m-1} & t_m \end{bmatrix} \cdot \begin{bmatrix} H(n+2,1) & H(n+1,1) & H(n,1) \\ H(n+2,0) & H(n+1,0) & H(n,0) \\ H(n+2,-1) & H(n+1,-1) & H(n,-1) \end{bmatrix}$$
$$= \begin{bmatrix} H(1,m+2) & H(0,m+2) & H(-1,m+2) \\ H(1,m+1) & H(0,m+1) & H(-1,m+1) \\ H(1,m) & H(0,m) & H(-1,m) \end{bmatrix} \cdot \begin{bmatrix} t_{n+3} & t_{n+2} & t_{n+1} \\ t_{n+2}+t_{n+1} & t_{n+1}+t_n & t_n+t_{n-1} \\ t_{n+2} & t_{n+1} & t_n \end{bmatrix}.$$

   And apply the fact that $H(n,-1) = H(-1,m) = 0$.

6. $H(n,m) = t_{n+1}t_{m+2} + it_{n+2}t_{m+1}$.

   Consider the (3,3) entry in

$$\begin{bmatrix} H(n+2,m+2) & H(n+1,m+2) & H(n,m+2) \\ H(n+2,m+1) & H(n+1,m+1) & H(n,m+1) \\ H(n+2,m) & H(n+1,m) & H(n,m) \end{bmatrix}$$
$$= \begin{bmatrix} t_{m+2} & t_{m+1}+t_m & t_{m+1} \\ t_{m+1} & t_m+t_{m-1} & t_m \\ t_m & t_{m-1}+t_{m-2} & t_{m-1} \end{bmatrix} \cdot \begin{bmatrix} 2+2i & 2+i & i \\ 1+2i & 1+i & i \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} t_{n+2} & t_{n+1} & t_n \\ t_{n+1}+t_n & t_n+t_{n-1} & t_{n-1}+t_{n-2} \\ t_{n+1} & t_n & t_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} t_{m+4} & t_{m+3}+t_{m+2} & t_{m+3} \\ t_{m+3} & t_{m+2}+t_{m+1} & t_{m+2} \\ t_{m+2} & t_{m+1}+t_m & t_{m+1} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} t_{n+4} & t_{n+3} & t_{n+2} \\ t_{n+3}+t_{n+2} & t_{n+2}+t_{n+1} & t_{n+1}+t_n \\ t_{n+3} & t_{n+2} & t_{n+1} \end{bmatrix}$$
$$= \begin{bmatrix} it_{m+3} & 0 & t_{m+4} \\ it_{m+2} & 0 & t_{m+3} \\ it_{m+1} & 0 & t_{m+2} \end{bmatrix} \cdot \begin{bmatrix} t_{n+4} & t_{n+3} & t_{n+2} \\ t_{n+3}+t_{n+2} & t_{n+2}+t_{n+1} & t_{n+1}+t_n \\ t_{n+3} & t_{n+2} & t_{n+1} \end{bmatrix}$$
$$= \begin{bmatrix} X & X & X \\ X & X & X \\ X & X & t_{m+2}t_{n+1} + it_{m+1}t_{n+2} \end{bmatrix}.$$

It could be more identities for this section. Also we can generalize $H(n,m)$ to any linear recurrence with constant coefficient of order 3 with some specific initial condition that mentioned in page 2 (page 145 in the book) of Pethe, [12].

6.4 Higher Dimensions

Consider 3 dimension analog of Harman from the same paper, [7].

Define $G(l, m, n)$ by

$$\begin{aligned} G(l+1, m, n) &= G(l, m, n) + G(l-1, m, n), \\ G(l, m+1, n) &= G(l, m, n) + G(l, m-1, n), \\ G(l, m, n+1) &= G(l, m, n) + G(l, m, n-1) \end{aligned}$$

where $G(a, b, c) = [a, b, c]$ for $a, b, c \in \{0, 1\}$.

Theorem 6.4.1 (This is an analog of (6) in previous section)**.**

$$G(l, m, n) = [F_l F_{m+1} F_{n+1}, F_{l+1} F_m F_{n+1}, F_{l+1} F_{m+1} F_n].$$

Proof. We could define a $2-2-2$ matrix which contains values of $G(l, m, n)$. Then the calculation is similar to (6) of previous section. □

References

[1] A. Benjamin and J. Quinn, *Proofs that really count*, Mathematical Association of America, 2003

[2] George Berzsenyi, *Gaussian Fibonacci Numbers*, The Fibonacci Quarterly 15(1977): 233-236.

[3] Bloom, D.M., 1996, *Problem 55*, Math Horizons (Sept.), 32.

[4] Bahar Demirturk and Refik Keskin, *Some new Fibonacci and Lucas identities by matrix methods* , International Journal of Mathematical Education in Science and Technology, 41:3, 379-387, October 2009.

[5] Bahar Demirturk, *Fibonacci and Lucas Sums by matrix methods* , International Mathematical Forum, 5, 2010, no.3, 99-107.

[6] S. Halici, *On Fibonacci Quaternions*, Advances in Applied Clifford Algebras, June 2012

[7] C. J. Harman, *Complex Fibonacci Numbers*, The Fibonacci Quarterly 19(1981): 82-86.

[8] Muthulakshmi R. Iyer, *Some Results on Fibonacci Quaternions*, The Fibonacci Quarterly, Volume 7, Number 2, April, 1969: 201-210.

[9] Robert C Johnson ,*Fibonacci numbers and matrices*, available at *maths.dur.ac.uk/∼dma0rcj/PED/fib.pdf*.

[10] J.H. Jordan, *Gaussian Fibonacci and Lucas Numbers*, The Fibonacci Quarterly, Volume 3, Number 4, December, 1965: 315-318.

[11] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley 2001

[12] S. Pethe, *Some identities For Tribonacci Sequences*, The Fibonacci Quarterly, Volume 26, Number 2, May, 1988: 144-151.