The Curious Bounds of Floor Function Sums

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Problem 1

For a fixed positive integer m, find the minimum and maximum of

$$S_m(\{a\}, K) = \sum_{k=0}^{K} \left(\left\lfloor \frac{a+k}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \right),$$

where $0 \leq a, K \leq m-1$.

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Answers:
It is clear that the minimum is 0 at (among other values of (a, K)) $a = 0$ and K can be any value.
The maximum is $m-1$ at $a = K = m-1$.

Problem 2

Define $S_m(\{a, b\}, K)$ as following:

$$S_m(\{a, b\}, K) := \sum_{k=0}^{K} \left(\left\lfloor \frac{a+b+k}{m} \right\rfloor - \left\lfloor \frac{a+k}{m} \right\rfloor - \left\lfloor \frac{b+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor \right),$$

where $0 \leq a, b, K \leq m - 1$

Find the minimum and maximum of $S_m(\{a, b\}, K)$.

Problem 2

Answers: The answers are not trivial:

$$0 \leq S_m(\{a, b\}, K) \leq \left\lfloor \frac{m}{2} \right\rfloor.$$

The minimum of S occurs when $(a = 0, b = 0, K = m - 1)$ or $(a + b + K < m)$ or $(a + b + K \geq 2m - 1)$. It was shown by multiple people: Carlitz, Grimson, Jacobsthal, and Tverberg.

The maximum was only shown recently by Thanatipanonda-Wong. The maximum of S occurs at $(a, b, K) = \left(\frac{m}{2}, \frac{m}{2}, \frac{m}{2} - 1\right)$, if m is even and at the combination of floor and ceiling of the above when m is odd.

Problem 3

Define $S_m(\{a,b,c\},K)$ as following:

$$S_m(\{a,b,c\},K) = \sum_{k=0}^{K} \Bigg(\left\lfloor \frac{a+b+c+k}{m} \right\rfloor - \left\lfloor \frac{a+b}{m} \right\rfloor - \left\lfloor \frac{b+c}{m} \right\rfloor - \left\lfloor \frac{a+c}{m} \right\rfloor + \left\lfloor \frac{a}{m} \right\rfloor + \left\lfloor \frac{b}{m} \right\rfloor + \left\lfloor \frac{c}{m} \right\rfloor - \left\lfloor \frac{k}{m} \right\rfloor \Bigg).$$

Find the minimum and maximum of $S_m(\{a,b,c\},K)$.

Problem 3

Answers:

$$-2\left\lfloor \frac{m}{2} \right\rfloor \leq S_m(\{a, b, c\}, K) \leq \left\lfloor \frac{m}{3} \right\rfloor.$$

The minimum of S occurs at $(a, b, c, K) = \left(\frac{m}{2}, \frac{m}{2}, \frac{m}{2}, \frac{m}{2} - 1\right)$, if m is even

and at the combination of the floor and ceiling of the above when m is odd. (Onphaeng-Pongsriiam, 2017)

The maximum of S occurs at $(a, b, c, K) = (n, n, n, n-1)$ and $(2n, 2n, 2n, 2n-1)$ if $m = 3n$. (Thanatipanonda-Wong)

Proof of the Minimum

The proof of the minimum is particularly nice.

Proof.

Notice the recursive relation:

$$S_m(\{a,b,c\},K) = S_m(\{a+b,c\},K) - S_m(\{b,c\},K) - S_m(\{a,c\},K).$$

Then by the result of problem 2:

$$\begin{aligned} 0 \le S_m(\{a+b,c\},K) &\le \left\lfloor \frac{m}{2} \right\rfloor, \\ -\left\lfloor \frac{m}{2} \right\rfloor \le -S_m(\{b,c\},K) &\le 0, \\ -\left\lfloor \frac{m}{2} \right\rfloor \le -S_m(\{a,c\},K) &\le 0. \end{aligned}$$

□

Proof of the Minimum (continued)

Proof. (continued).

After adding up all these bounds we have that

$$-2\left\lfloor \frac{m}{2} \right\rfloor \leq S_m([a,b,c],K)$$

This bound is also sharp since $(a,b,c,K) = \left(\frac{m}{2}, \frac{m}{2}, \frac{m}{2}, \frac{m}{2} - 1\right)$ gives the minimum to each of the three inequalities. □

The General Case

The general case can be stated as follows:

$$S_m(\{a_1, ..., a_n\}, K) = \sum_{k=0}^{K} \sum_{T \subset [1,n]} (-1)^{n-|T|} \left\lfloor \frac{k + \sum_{i \in T} a_i}{m} \right\rfloor .$$

Known Results

For odd n, $n \geq 3$:

$$-2^{n-2}\left\lfloor \frac{m}{2} \right\rfloor \leq S_m(\{a_1, \ldots, a_n\}, K).$$

For even n, $n \geq 2$:

$$S_m(\{a_1, \ldots, a_n\}, K) \leq 2^{n-2}\left\lfloor \frac{m}{2} \right\rfloor.$$

The extreme value of each case is attained at

$$A = \{m/2, m/2, \ldots, m/2\}, \quad K = m/2 - 1.$$

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These results were shown by Onphaeng-Pongsriiam (2017) using the recursive method we did earlier.

The Other Missing Half?

The other missing half, for $n \geq 4$, are still open and has been conjectured by Thanatipanonda-Wong.

These are the conjectures for $n = 4, 5$.

$n = 4$:

$$-3 \cdot \left\lfloor \frac{m}{3} \right\rfloor \leq S_m$$

$n = 5$:

$$S_m \leq 6 \cdot \left\lfloor \frac{m}{3} \right\rfloor$$

In both cases, the extreme values occur at:

$$A = \{m/3, m/3, ..., m/3\}, K = m/3 - 1$$
$$\text{or } A = \{2m/3, 2m/3, ..., 2m/3\}, K = 2m/3 - 1.$$