TWO GAMES ON ARITHMETIC FUNCTIONS: SALIQUANT AND NONTOTIENT

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Abstract. We investigate the Sprague-Grundy sequences for two normal-play impartial games based on arithmetic functions, first described by Iannucci and Larsson in [IL]. In each game, the set of positions is $\mathbb{N}$. In SALIQUANT, the options are to subtract a non-divisor. Here we obtain several nice number theoretic lemmas, a fundamental theorem, and two conjectures about the eventual density of Sprague-Grundy values.

In NONTOTIENT, the only option is to subtract the number of relatively prime residues. Here are able to calculate certain Sprague-Grundy values, and start to understand an appropriate class function.

1. Introduction

In this paper, we study two of the games introduced by [IL]. Their rules are as follows.

(a) Saliquant. Subtract a non-divisor: For $n \geq 1$, $\text{opt}(n) = \{n - k : 1 \leq k \leq n : k \nmid n\}$.
(b) Nontotient. Subtract the number of relatively prime residues: For $n \geq 1$, $\text{opt}(n) = \{n - \phi(n)\}$, where ϕ is Euler's totient function.

In each case, we examine the normal-play variant only, so the usual Sprague-Grundy theory applies. In particular, the *nim-value* of a position n is recursively given by

$$\mathcal{SG}(n) = \text{mex}\{\mathcal{SG}(x) \mid x \in \text{opt}(n)\},$$

where $\text{mex}(A)$ is the least nonnegative integer not appearing in A. Chapter 7 of [LIP] gives a readable overview for the newcomer. Note that for games of no choice, such as NONTOTIENT, $\mathcal{SG}(n)$ calculates the parity of the number of moves required to reach a terminal position. The sole terminal position for NONTOTIENT is 1.

2. Let's play saliquant!

Inaucci and Larsson give a uniform upper bound for nim-values of SALIQUANT positions and show that odd positions attain this bound:

Lemma 2.1 ([IL],Theorem 4)**.** *In* SALIQUANT*,*

- *If n is odd, then $\mathcal{SG}(n) = \frac{n-1}{2}$*
- *For all $n \geq 1$, $\mathcal{SG}(n) < \frac{n}{2}$*

Our task, therefore, will be to investigate the nim-values of even positions. The first few such values are:

n	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42
$\mathcal{SG}(n)$	0	1	1	3	2	4	6	7	4	7	5	10	12	10	13	15	8	13	9	17	17

First we can establish some particular cases where the nim-value will be below the uniform upper bound given in the last part of the Lemma:

Lemma 2.2.

- *If $3 \mid n$, then $\mathcal{SG}(2n) \leq n - 2$.*
- *If $5 \mid n$, then $\mathcal{SG}(4n) \leq 2n - 3$.*

Proof. If $3 \mid n$, then $2n - 4$ is the largest possible option of $2n$. So by Lemma 2.1 all options have a nim-value of at most $n - 3$. Hence $\mathcal{SG}(2n) \leq n - 2$.

Similarly, if $5 \mid n$, then $4n - 6$ is the largest possible option of $4n$, with the exception of $4n - 3$. So all options have a nim-value of at most $2n - 4$, or exactly $2n - 2$. Hence $\mathcal{SG}(4n) \leq 2n - 3$. □

Note that the former bound is sharp. For example, setting $n = 15$, we see that $\mathcal{SG}(30)$ is 13. Next we establish a uniform lower bound.

Lemma 2.3. *If p is the smallest prime divisor of n, then $\mathcal{SG}(n) \geq \mathcal{SG}(\frac{p-1}{p}n)$; in particular, $\mathcal{SG}(2n) \geq \mathcal{SG}(n)$.*

Proof. Let $n-k < \frac{p-1}{p}n$, where p is the smallest prime divisor of n. Then $\frac{n}{p} < k < n$, and so $k \nmid n$. Hence $n-k$ is an option of n. n has every option which $\frac{p-1}{p}n$ has. Thus, $\mathcal{SG}(n) \geq \mathcal{SG}(\frac{p-1}{p})n$. □

Corollary 2.4. $\mathcal{SG}(n) \geq \frac{n-2}{4}$ *for all n.*

Proof. Lemma 2.1 establishes this for odd n.

Next if $n = 2k$, where k is odd, then Lemma 2.3 tells us that $\mathcal{SG}(n) \geq \mathcal{SG}(k) = \frac{k-1}{2} = \frac{n-2}{4}$.

Now let $n = 2^m k$, where k is odd and $m \geq 2$. Then $n-(k+2), n-(k+4), \ldots, 1$ are all options of n, with nim-values $\frac{1}{2}(n-k-3), \frac{1}{2}(n-k-5), \ldots, 0$, respectively. Thus

$$\mathcal{SG}(n) \geq \frac{1}{2}(n-k-1) \geq \frac{1}{2}\left(n - \frac{n}{4} - 1\right) > \frac{1}{2}\left(\frac{n}{2} - 1\right) > \frac{n-2}{4}$$

□

Next we prove a key lemma about the nim-values of even positions.

Lemma 2.5. *If $\mathcal{SG}(2n) = n-k$, then $2k-1 \mid n$.*

Proof. Suppose $\mathcal{SG}(2n) = n-k$. Then $2n$ has no option of nim-value $n-k$. Since $\mathcal{SG}(2n-2k+1) = n-k$, it is not an option. In other words, $2k-1 \mid 2n$ and hence $2k-1 \mid n$. □

From here, we can establish nim-values of several particular cases of even numbers. To start, the previous result immediately narrows the possibilities of double a prime or semiprime.

Corollary 2.6. *Let p, q be odd primes, then*

- $\mathcal{SG}(2p) = p-1$ *or* $p - \frac{p+1}{2} = \frac{p-1}{2}$*; and*
- $\mathcal{SG}(2pq) = pq-1, pq - \frac{p+1}{2}, pq - \frac{q+1}{2}$*, or* $pq - \frac{pq+1}{2} = \frac{pq-1}{2}$.

We next refine the first bullet point.

Lemma 2.7. *Let $p \geq 5$ be prime. Then the only possible option of $2p$ with nim-value $\frac{p-1}{2}$ is $p+1$. Hence*

- $\mathcal{SG}(p+1) = \frac{p-1}{2} \implies \mathcal{SG}(2p) = p-1$
- $\mathcal{SG}(p+1) \neq \frac{p-1}{2} \implies \mathcal{SG}(2p) = \frac{p-1}{2}$.

Note that if $p = 3$, then $p+1 = 4$ is not an option of $2p = 6$.

Proof. Let $x \in \mathrm{opt}(2p)$ such that $\mathcal{SG}(x) = \frac{p-1}{2}$.

By Lemma 2.1, if x were odd, then we would have $x = p$, but p is not an option of $2p$. So let $x = 2n$ for some n. By Lemma 2.5, since $\mathcal{SG}(2n) = \frac{p-1}{2} = n - (n - \frac{p-1}{2})$, we have $2(n - \frac{p-1}{2}) - 1 \mid n$, and so $2n - p \mid n$. On one hand, this implies that $n < p$.

On the other hand, it means that we can write $n = d(2n-p) = 2dn - dp$ for some $d \in \mathbb{N}$. Then $n \mid dp$ and $d \mid n$. Now since $n < p$ and p is prime, we have $n \mid d$. Finally, since $d \mid n$, this means $n = d$, and so $2n - p = 1$ or $x = 2n = p+1$. □

In fact, this is enough to generate infinitely many examples for which our uniform lower bound is attained.

Theorem 2.8. *If p is prime and $p \equiv 5 \bmod 6$, then $\mathcal{SG}(2p) = \frac{p-1}{2}$.*

Proof. Let p be prime where $p \equiv 5 \bmod 6$. We claim that $\mathcal{SG}(p+1) \neq \frac{p-1}{2}$, and so the previous lemma implies that $\mathcal{SG}(2p) = \frac{p-1}{2}$.

Indeed, since $p \equiv 5 \bmod 6$, we have $p+1 \equiv 0 \bmod 6$. In particular, $1, 2, 3 \mid p+1$, so the largest possible option of $p+1$ is $(p+1) - 4 = p-3$. So by Lemma 2.1, for all $y \in \mathrm{opt}(p+1)$, $\mathcal{SG}(y) < \frac{y}{2} \leq \frac{p-3}{2}$, that is $\mathcal{SG}(y) \leq \frac{p-5}{2}$. Hence $\mathcal{SG}(p+1) \leq \frac{p-3}{2}$. □

Corollary 2.9. *There are infinitely many $n \in \mathbb{N}$ such that $\mathcal{SG}(n) = \frac{n-2}{4}$.*

Proof. It is well known that there are infinitely many primes $p = 5 \bmod 6$. For each of these p, letting $n = 2p$, we have $\mathcal{SG}(n) = \mathcal{SG}(2p) = \frac{p-1}{2} = \frac{n-2}{4}$. □

It is possible to keep refining this inquiry about numbers which are twice an odd. For example Corollary 2.6 could be extended for more than 2 odd prime factors, but we don't see how helpful it is. Instead, we investigate the remaining cases by decomposing even numbers as an odd number times a power of 2. As a first step, we can compute exact nim-values in the case that the odd part is 1, 3, 5, or 9.

Lemma 2.10. *Let $b \geq 1$. Then $\mathcal{SG}((2a+1)2^b) = (2a+1)2^{b-1} - a - 1$ for $a = 0, 1, 2, 4$.*

Proof. This can be checked by hand for the cases when $b = 1$ or $b = 2$, so let $b \geq 3$, and consider the options of $(2a+1)2^b$

All odd numbers greater than $(2a+1)$ are non-divisors of $(2a+1)2^b$, so the odd numbers $1, 3, \ldots, (2a+1)2^b-(2a+3)$ are all options with nim-values $0, 1, \ldots, (2a+1)2^{b-1}-a-2$, respectively.

We claim that there is no option with nim-value $(2a+1)2^{b-1} - a - 1$. Indeed $(2a+1)2^b - 2a - 1$ is not an option, and is the only odd number with nim-value $(2a+1)2^{b-1} - a - 1$. Next, note that $b \geq 3$ and $a = 0, 1, 2, 4$ i.e. $(2a+1) = 1, 3, 5, 9$. Hence all even numbers less than $(2a+1)$ divide $(2a+1)2^b$ and the only even options are less than or equal to $(2a+1)2^b - 2a - 2$. By Lemma 2.1, their nim-values are less than $\frac{(2a+1)2^b-2a-2}{2} = (2a+1)2^{b-1} - a - 1$. Thus, there is no option with nim-value $(2a-1)2^{b-1} - a - 1$, and $\mathcal{SG}((2a-1)2^b) = (2a-1)2^{b-1} - a - 1$. □

We now see that there are infinitely many *even* values for which our uniform upper bound is obtained:

Corollary 2.11. *Let $b \geq 1$. Then $\mathcal{SG}(2^b) = 2^{b-1} - 1$. In particular, there are infinitely many m for which $\mathcal{SG}(n) = \frac{n-2}{2}$.*

Note that the above proof does not work, for example, when $a = 3$ i.e. $2a + 1 = 7$, since $6 < 2a + 1$, and $6 \nmid 7(2^b)$. In fact $\mathcal{SG}(14) = 6$, not $(2a+1)2^{b-1} - a - 1 = 3$. Next we obtain a slightly weaker result when $a = 10$ and $2a + 1 = 21$.

Lemma 2.12. *Let $b \geq 1$. Then $\mathcal{SG}(21(2^b)) = 21(2^{b-1}) - 11$ or $21(2^{b-1}) - 4$.*

Proof. In the case $b = 1$, we see $\mathcal{SG}(42) = 17$. For $b \geq 2$, consider the options of $21(2^b)$. The odd numbers $1, 3, \ldots, 21(2^b) - 23$ and $21(2^b) - 19, \ldots, 21(2^b) - 9$ are all options with nim-values $0, 1, \ldots, 21(2^{b-1}) - 12$ and $21(2^{b-1}) - 10, \ldots, 21(2^{b-1}) - 5$, respectively. The numbers $21(2^b) - 21$ and $21(2^b) - 7$ with nim-values $21(2^{b-1}) - 11$ and $21(2^{b-1}) - 4$ are not options, and all larger odd numbers have nim-values greater than $21(2^{b-1}) - 4$.

On the other hand, since $2, 4, 6 \mid 21(2^b)$, Lemma 2.1 implies that any even options have nim-values less than $\frac{21(2^b)-8}{2} = 21(2^{b-1}) - 4$. Hence $\mathcal{SG}(21(2^b)) = 21(2^{b-1}) - 11$ or $21(2^{b-1}) - 4$. □

We end this section by showing that twice a Mersenne number is above the uniform lower bound. Note that if $m = 2n = 2(2^b - 1)$ then $\frac{m-2}{4} = \frac{n-1}{2} = 2^{b-1} - 1$.

Lemma 2.13. *Let $b \geq 3$. Then $\mathcal{SG}(2(2^b - 1)) > 2^{b-1} - 1$. In particular, if $2^b - 1$ is prime, then $\mathcal{SG}(2(2^b - 1)) = 2^b - 2$.*

Proof. By Corollary 2.11, $\mathcal{SG}(2^b) = 2^{b-1} - 1$, so we just need to show that $2^b \in \operatorname{opt}(2(2^b - 1))$.

Suppose otherwise and that $2(2^b - 1) - 2^b \mid 2(2^b - 1)$. Then $2^b - 2 \mid 2(2^b - 1)$. Thus either $2^b - 2$ and $2^b - 1$ share a common factor and so $2^b - 2 = 1$, or $2^b - 2 \mid 2$ and so $b \leq 2$. Both cases are impossible.

In the case $2^b - 1$ is prime, Corollary 2.6 implies $\mathcal{SG}(2(2^b - 1)) = 2^b - 2$. □

3. The Fundamental Theorem of saliquant and density of values

Finally, we obtain our most general statement about nim-values of Saliquant. The two corollaries which follow were actually proved first, inspired by the proof of Corollary 2.4.

Theorem 3.1. *For all $a \geq 0, b \geq 1$,*

$$\begin{aligned}\mathcal{SG}\left((2a+1)2^b\right) &= \frac{m}{2m+1}\left((2a+1)2^b - 1\right) + \frac{1}{2m+1}\left((2a+1)2^{b-1} - a - 1\right) \\ &= (2a+1)2^{b-1} - \frac{1}{2}\left(\frac{2a+1}{2m+1} + 1\right)\end{aligned}$$

for some non negative integer m. Thus

$$\mathcal{SG}\left((2a+1)2^b\right) = (2a+1)2^{b-1} - \frac{d+1}{2}, \text{ where } d \text{ is a factor of } 2a+1.$$

This theorem unifies several edge cases, as well. If we set $a = 0$, then we must have $d = 1$, obtaining Corollary 2.11. Let $f(a,b,m)$ be the function given by Theorem 3.1. If we set $b = 0$, then $f(a,b,m)$ is never an integer, but $\lim_{m\to\infty} f(a,b,m) = \frac{n-1}{2}$, matching Lemma 2.1.

Fixing a and b, $f(a,b,m)$ is a linear rational function in m, thus monotonic for $m \geq 0$, and it is easily checked that it is increasing. Hence its minimum is obtained when $m = 0$, with an upper bound given by $m \to \infty$. Thus we have the following corollary, which itself is a generalization of Lemma 2.10.

Corollary 3.2. *For all $a, b \geq 1$,*

$$\frac{(2a+1)2^b}{2} - a - 1 \leq \mathcal{SG}\left((2a+1)2^b\right) < \frac{(2a+1)2^b}{2} - \frac{1}{2}.$$

The upper bound is the same as in Lemma 2.1. If we fix a and let b grow large, the lower bound is an asymptotic improvement over Corollary 2.4 from $\mathcal{O}(\frac{n}{4})$ to $\mathcal{O}(\frac{n}{2})$. Furthermore, we will see experimentally below that all values of $f(a,b,m)$ are obtained. To illustrate the theorem, set $b = 1$ to obtain all possible nim-values of even numbers which are not multiples of 4:

Corollary 3.3. *For all $a \geq 1$, $\mathcal{SG}(4a+2)$ must have the form*

$$\frac{(4m+1)a+m}{2m+1} \qquad \left(= a, \frac{5a+1}{3}, \frac{9a+2}{5}, \frac{13a+3}{7}, \frac{17a+4}{9}, \frac{21a+5}{11}, \ldots\right)$$

for some $m \geq 0$.

Proof of Theorem 3.1. Suppose $a, b \geq 1$. Let $X = \mathcal{SG}\left((2a+1)2^b\right)$. Then

$$X = ((2a+1)2^{b-1}) - ((2a+1)2^{b-1} - X),$$

so by Lemma 2.5, we have

$$\left(2\left((2a+1)2^{b-1} - X\right) - 1\right) \mid (2a+1)2^{b-1}.$$

Thus there is some Q_1 so that

$$Q_1(a2^{b+1} + 2^b - 2X - 1) = (2a+1)2^{b-1}.$$

Since $(a2^{b+1} + 2^b - 2X - 1)$ is odd, $2^{b-1} \mid Q_1$. Pick Q_2 so that $Q_2 2^{b-1} = Q_1$. This gives

$$Q_2(a2^{b+1} + 2^b - 2X - 1) = 2a+1.$$

Next since Q_2 is odd, we can set $Q_2 = 2m+1$ for some $m \geq 0$, giving

$$(2m+1)(a2^{b+1} + 2^b - 2X - 1) = 2a+1.$$

Finally, solving for X gives the desired result. □

Now that we know the specific possible values $\mathcal{SG}(n)$ can take based on the decomposition $n = (2a+1)2^b$, a natural question is how these values are distributed. For a given $b > 0$, $m \geq 0$, define

$$S_{b,m} = \{a \in \mathbb{N} \mid \mathcal{SG}((2a+1)2^b) = f(a,b,m)\}.$$

The experimental density of $S_{b,m}$ for $b = 1,2,3,4$ and $m = 0,1,2,3,4$ are shown in Table 1. For $b = 1$, we measured up to $a = 5000$; for $b = 2,3$, up to $a = 2000$; and for $b = 4$, up to $a = 1000$. The associated Maple program can be found at the third author's website `http://www.thotsaporn.com`.

In Figure 1, we can see some of these values, with the corresponding labels given in Table 1. For example, consider the entry of the table marked **(C)**. It says that the density of numbers of the form $x = 8a+4$ for which $\mathcal{SG}(x) = 3a+1$ is 0.561. Then we can see that the line in the figure with slope $\frac{3}{8}$ (also marked **(C)**) has about half density. Contrast with the entry marked **(D)**, corresponding to the line with slope $\frac{5}{12}$. It is very sparse, as seen in the figure. Notice that the y-intercept of each of these lines corresponds to $a = -\frac{1}{2}$, which in each case gives

$$f\left(-\frac{1}{2}, b, m\right) = \frac{1}{2m+1}\left(-m2^b - 2^{b-1} + \frac{1}{2} + m2^b - m + 2^{b-1} - 1\right) = \frac{-m-\frac{1}{2}}{2m+1} = -\frac{1}{2},$$

which is ok to be negative, since the game is only meaningfully defined on positive numbers. Finally, the line marked **(A)** is $y = \frac{x-1}{2}$, which includes all odd x and some even x, per Lemma 2.1 and Corollary 2.11.

Table 1. Experimental values of $S_{b,m}$. The labels **(B)**—**(E)** match Figure 1.

m	$\mathcal{SG}(4a+2)$ $= f(a,1,m)$	density ($a \leq 5000$)	$\mathcal{SG}(8a+4)$ $= f(a,2,m)$	density ($a \leq 2000$)	$\mathcal{SG}(16a+8)$ $= f(a,3,m)$	density ($a \leq 2000$)	$\mathcal{SG}(32a+16)$ $= f(a,4,m)$	density ($a \leq 1000$)
0	a **(B)**	0.532	$3a+1$ **(C)**	0.561	$7a+3$ **(E)**	0.540	$5a+7$	0.638
1	$\frac{5a+1}{3}$ **(D)**	0.026	$\frac{11a+4}{3}$	0.056	$\frac{23a+10}{3}$	0.090	$\frac{47a+22}{3}$	0.069
2	$\frac{9a+2}{5}$	0.037	$\frac{19a+7}{5}$	0.044	$\frac{39a+17}{5}$	0.050	$\frac{79a+37}{5}$	0.046
3	$\frac{13a+3}{7}$	0.061	$\frac{27a+10}{7}$	0.049	$\frac{55a+24}{7}$	0.046	$\frac{111a+52}{7}$	0.043
4	$\frac{17a+4}{9}$	0.022	$\frac{35a+13}{9}$	0.030	$\frac{71a+31}{9}$	0.015	$\frac{143a+67}{9}$	0.010

We next show a straightforward upper bound for these densities, noting that each of the values in Table 1 are well below this bound.

Lemma 3.4. *Given $b \geq 1$, $m \geq 0$, the density of $S_{b,m}$ is at most $\frac{1}{2m+1}$.*

Proof. Fix $b \geq 1$, $m \geq 0$, and consider

$$\begin{aligned}(2m+1)f(a,b,m) &= \left(a(m2^{b+1} + 2^b - 1) + m(2^b - 1) + (2^{b-1} - 1)\right) \\ &= a\left((2m+1)2^b - 1\right) + (2m+1)2^{b-1} - m - 1 \\ &\equiv -a - m - 1 \bmod (2m+1)\end{aligned}$$

Thus $f(a,b,m)$ is only an integer when $a \equiv m \bmod (2m+1)$, and so $\frac{1}{2m+1}$ is an upper bound for how frequently $\mathcal{SG}((2a+1)2^b)$ can attain this value. □

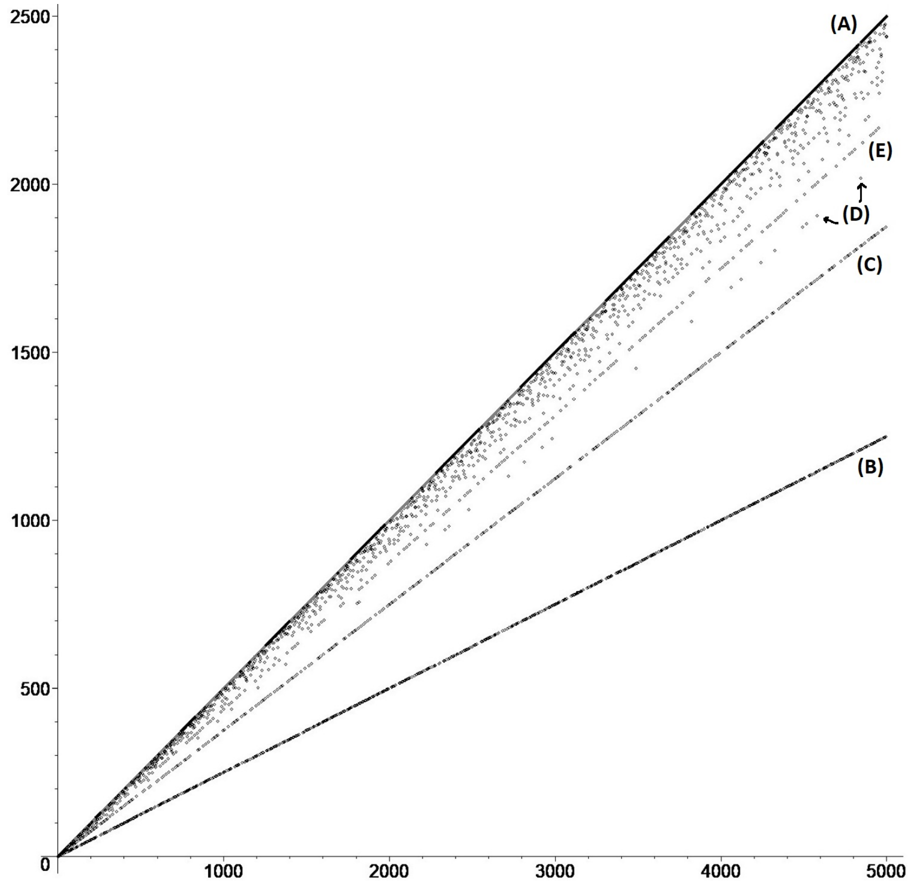

FIGURE 1. The first 5000 nim-values of SALIQUANT. The slopes of the labelled lines are **(A)** $\frac{1}{2}$, **(B)** $\frac{1}{4}$, **(C)** $\frac{3}{8}$, **(D)** $\frac{5}{12}$, **(E)** $\frac{7}{16}$

Given the values in Table 1, we suspect that most of these values are actually 0:

Conjecture 1. *For a given $b > 0$,*

- *If $m = 0$, $S_{b,m}$ has positive density, and*
- *If $m > 0$, $S_{b,m}$ has density 0, but is nonempty.*

We can also look at how fixing a and b affects the value of m. Define $M(a,b) = m$ where $\mathcal{SG}((2a+1)2^b) = f(a,b,m)$, and consider Table 2.

TABLE 2. Experimental values of $M(a,b)$.

	$a=3$	$a=4$	$a=5$	$a=6$	$a=7$	$a=8$	$a=9$	$a=10$	$a=11$
$b=1$	3	0	0	6	2	0	0	1	0
$b=2$	0	0	0	0	0	8	0	1	0
$b=3$	0	0	0	6	0	0	9	0	0
$b=4$	0	0	0	0	0	0	0	0	11
$b=5$	0	0	0	0	0	0	0	0	0
$b=6$	0	0	0	0	0	0	9	0	0
$b=7$	0	0	0	0	0	0	0	0	0
$b=8$	0	0	0	0	0	0	0	0	0
$b=9$	0	0	0	0	0	0	0	0	0
$b=10$	0	0	0	0	0	0	0	0	0

If one imagines running along any row of Table 2 and tracking the distribution of m, they would get the densities achieved in Table 1 as a tends toward infinity. For our second conjecture, we instead consider the behavior of m rather than of a, noting that it is difficult to generate more data as the values grow exponentially as b increases.

Given the sparsity of each column, we suspect that in each column all but a finite number of values are non-zero.

Conjecture 2. *For a given $a > 0$, for sufficiently large b, $M(a, b) = 0$, in which case* $\mathcal{SG}((2a + 1)2^b) = (2a + 1)2^{b-1} - a - 1$.

Note that Lemma 2.10 proves a stronger form of the conjecture for $a = 0, 1, 2, 4$, and Lemma 2.12 shows that when $a = 10$ we have either $m = 0$ or $m = 1$.

4. NONTOTIENT

Denoting $\phi(n) = |\{1 \leq k \leq n \mid k \text{ is not a factor of } n\}|$, [IL] also define two games based on $\phi(n)$:

- Totient: $\text{opt}(n) = \phi(n)$
- Nontotient: $\text{opt}(n) = n - \phi(n)$

In this section we make some headway in understanding nontotient. First recall that $\phi(ab) = \phi(a)\phi(b)$, and for prime p, $\phi(p^k) = p^{k-1}(p - 1)$. Thus if $n = p_1^{k_1} \dots p_m^{k_m}$, we have $\phi(n) = \prod {p_i}^{k_i-1}(p_i - 1)$. In particular $\phi(1) = 1$. Define $g(n) := \text{opt}(n) = n - \phi(n)$. We immediately obtain:

Lemma 4.1. *For $n > 2$, $\phi(n)$ is even, and so $g(n)$ has the same parity as n.*

For the rest of the section, let p and q always represent primes. As noted in [IL], $g(p^k) = p^{k-1}$. Hence the game on p^k terminates after k moves and so $\mathcal{SG}(p^k) = 0$ if and only if k is even. They also note that $g(p^k q) = p^{k-1}(q + p - 1)$, and so in the case that $q + p - 1$ is a power of p, this becomes easy to compute. Consider for example the prime pairs $(p, q) = (2, 7)$ or $(3, 7)$. We can extend this as follows. First note that

$$g(p^k q^l) = p^k q^l - p^{k-1}(p - 1)q^{l-1}(q - 1) = p^{k-1}q^{l-1}(p + q - 1).$$

Then we have

Theorem 4.2.

(a) If $q = p^b - p + 1$ where b is even, then $\mathcal{SG}(p^k q^l) = 0$ if and only if q is even.
(b) If $q = p^b - p + 1$ where b is odd, then $\mathcal{SG}(p^k q^l) = 0$ if and only if $q + l$ is even.

Proof. In this case $g(p^k q^l) = p^{k-1}q^{l-1}\left(p + (p^b - p + 1) - 1\right) = p^{k+b-1}q^{l-1}$. So after l moves, the position will be $p^{k+l(b-1)}$, and thus the game terminates after $k + l(b - 1) + l = k + lb$ moves. □

Some prime pairs (p, q) that satisfy part (a) are $(2, 3)$, $(3, 7)$, $(7, 43)$, $(13, 157)$, $(3, 79)$, $(11, 14631)$, $(3, 727)$. For part (b) we have $(2, 7)$, $(7, 337)$, and $(19, 2476081)$. Part (b) also applies to each pair $(2, 2^p - 1)$ for each Mersenne prime $2^p - 1$. As a next step, one might analyze cases which reduce to one of the above cases in a predictable number of steps. For example

Corollary 4.3. *$\mathcal{SG}(2^k 5) = 0$ if and only if k is odd.*

Proof. Here $g(2^k 5) = 2^{k-1}(6) = 2^k 3$, and so the result follows by Theorem 4.2 (a). □

The authors of [IL] were able to use Harold Shapiro's height function, $H(n) = H(\phi(n)) + 1$, to give a method for computing the nim-value of any natural number in totient. Motivated by this success, they suggest analyzing a class function $\text{dist}(n) = i$, which gives the least i for which $g^i(n)$ is a prime power. We instead analyze the function $C(n) = i$ if $g^i(n) = 1$. The initial values are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$C(n)$	0	1	1	2	1	3	1	3	2	4	1	4	1	4	2	4	1	5	1	5

Lemma 4.4. $C(4n) = C(2n) + 1$.

Proof. Note that for $k > 1$ and m odd, $\phi(2^k m) = \phi(2^k)\phi(m) = 2^{k-1}\phi(m) = 2\phi(2^{k-1}m)$. Hence we have $g(4n) = 4n - \phi(4n) = 4n - 2\phi(2n) = 2(2n - \phi(2n)) = 2g(2n)$. So if $g^i(2n) = 1$, then $g^i(4n) = 2$, which means $g^{i+1}(4n) = 1$. □

Corollary 4.5. $\mathcal{SG}(4n) = 1 - \mathcal{SG}(2n)$.

For example, knowing that $\mathcal{SG}(10) = 0$, we again obtain Corollary 4.3. We end this section with some observations about the function $C(n)$ for even n.

Lemma 4.6. *If n is even and $2^{i-1} < n \leq 2^i$, then $C(n) \geq i$.*

The least value we don't see equality is $C(30) = 6$.

Proof. We proceed by induction on i, observing initial cases in the table above. Suppose $2^{i-1} < n \leq 2^i$ and $n = 2k$. Then $\phi(n) \leq k$, so $g(n) \geq k > 2^{i-2}$. By Lemma 4.1 $g(n)$ is also even, so we have by induction that $C(g(n)) \geq i - 1$. Hence $C(n) \geq i$. □

Lemma 4.7. *If p is an odd prime, then $C(2p) = C(2(p + 1))$.*

Proof. We have $g(2p) = p + 1$, so $C(2p) = C(p + 1) + 1 = C(2(p + 1))$, by Lemma 4.4. □

The first time the conclusion does not hold is when $p = 15$, since $C(30) = 6$ and $C(32) = 5$.

Theorem 4.8. *Let $i \geq 1$. The set $S_i = \{C(n) \mid 2^{i-1} \leq n \leq 2^i$ and n is even$\}$ is an interval of $\mathbb{N}$.*

Proof. We proceed by induction, again noting initial values in the chart above. For each $i \geq 1$, Lemma 4.6 implies that the minimal possible value of S_i is i, and this is in fact obtained by $C(2^i)$.

Now let $i \geq 2$, and suppose that the maximal value in S_{i-1} is M. By induction $S_{i-1} = \{i - 1, i, \ldots, M\}$. Lemma 4.4 then implies that $\{i, i + 1, \ldots, M + 1\} \subseteq S_i$.

Next suppose that $\{i, i + 1, \ldots, M + 1\} \neq S_i$. Then there is some even $2^{i-1} \leq y \leq 2^i$ for which $C(y) > M + 1$. In this case, since $g(y)$ is even and $C(g(y)) > M$, we must have $2^{i-1} \leq g(y)$. Thus $C(g(y)) = C(y) - 1 \in S_i$. □

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