Moment Calculus

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Contents

1 Introduction

Dr. Z.

- Continuous and discrete paradigms: $\epsilon - \delta$ Cauchy-Weierstrass-style.
- Complex analysis was a different story. It has a feel of discrete, we use power series and a formal power series.
- On a fundamental level, continuous mathematics is just a degenerate case of the discrete, as I will show you today.

Mine

- Discrete math. is more natural and down-to-earth.
- Probability is especially neat to me as it is not just counting stuff but have some meaning.
- This work from my advisor shows the connection between discrete and continuous mathematics.
- New algorithm for automated proof.

2 Moment Calculus and Central Limit Theorem

Inverse-Fourier-Transform: $\sum_r \frac{m_r(it)^r}{r!} = E[e^{itX}]$.

2.1 Basic probability

We have a *finite* set S, called the sample space consisting of simple events, s. Each s has a probability p_s attached to it where $\sum_{s\in S} p_s = 1$.

We also have a *random variable* $X : S \to R$, where R is a finite set of real numbers. We are interested in its probability distribution $Pr(s \in S | X(s) = r)$.

Next, the first moment,

$$\mu = E[X] := \sum_{s\in S} p_s X(s).$$

Analogously, the *higher moments* (about the mean) are defined by

$$m_r(X) = E[(X-\mu)^r] := \sum_{s\in S} p_s (X(s) - \mu)^r.$$

Note $m_1 = 0$.

2.2 Moment Generating Function

$$\phi(t) = E[e^{tX}] = \begin{cases} \sum_x e^{tx} p(x) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & \text{if } x \text{ is continuous.} \end{cases}$$

Note $\phi^n(0) = E[X^n]$.

Moment Generating Function of standard normal distribution

$$\phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}} dx = e^{\frac{t^2}{2}}.$$

Moment Generating Function of binomial distribution

We will show binomial distribution is asymptotically normal.

$$\begin{aligned}
\phi(t) &= \sum_{x=0}^{n} \binom{n}{x} e^{t\left(\frac{x-np}{\sqrt{np(1-p)}}\right)} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} \binom{n}{x} \left(pe^{\frac{t(1-p)}{\sqrt{np(1-p)}}}\right)^x \left((1-p)e^{\frac{-tp}{\sqrt{np(1-p)}}}\right)^{n-x} \\
&= \left(pe^{\frac{t(1-p)}{\sqrt{np(1-p)}}} + (1-p)e^{\frac{-tp}{\sqrt{np(1-p)}}}\right)^n \\
&\approx \left(p\left[1 + \frac{t(1-p)}{\sqrt{np(1-p)}} + \frac{t^2(1-p)^2}{2np(1-p)}\right] + (1-p)\left[1 - \frac{tp}{\sqrt{np(1-p)}} + \frac{t^2p^2}{2np(1-p)}\right]\right)^n \\
&= \left(1 + \frac{t^2}{2n}\right)^n \\
&= e^{\frac{t^2}{2}} \qquad \text{as } n \to \infty.
\end{aligned}$$

2.3 Central Limit Theorem

Standard Normal Distribution:

$$P(a \le X \le b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx$$

Central Limit Theorem:

Let X_k be a sequence of mutually independent random variables with a common distribution. Suppose that $\mu := E[X_k]$ and $\sigma^2 := Var[X_k]$ exist and let $S_n = X_1 + X_2 + \cdots + X_n$ Then for every fixed β,

$$P\{\frac{S_n - n\mu}{\sigma\sqrt{n}} < \beta\} \to \mathcal{N}(\beta).$$

Proof. Let $Y_n := \dfrac{X_n - \mu}{\sigma}$.
Clearly Y_n has mean 0 and standard deviation 1.

Consider moment generating function of $\dfrac{Y_1 + Y_2 + \cdots + Y_n}{\sqrt{n}}$

$$\begin{aligned}
E[e^{t\left(\frac{Y_1+\cdots+Y_n}{\sqrt{n}}\right)}] &= E[e^{\frac{tY_1}{\sqrt{n}}} e^{\frac{tY_2}{\sqrt{n}}} \ldots e^{\frac{tY_n}{\sqrt{n}}}] \\
&= E[e^{\frac{tY}{\sqrt{n}}}]^n \qquad \text{by independence} \\
&\approx E\left[1 + \frac{tY}{\sqrt{n}} + \frac{t^2Y^2}{2n}\right]^n \\
&= \left[1 + \frac{t}{\sqrt{n}} E[Y] + \frac{t^2}{2n} E[Y^2]\right]^n \\
&= \left[1 + \frac{t^2}{2n}\right]^n \\
&= e^{\frac{t^2}{2}} \qquad \text{as} \quad n \to \infty.
\end{aligned}$$

which is the moment generating function of a standard normal distribution. □

2.4 Connection between moment m_r and moment generating function

$$\phi(t) := E[e^{tX}] = 1 + tm_1 + \frac{t^2 m_2}{2!} + \ldots.$$

This implies $\phi^{(n)}(0) = m_n$.

Moment of the standard normal distribution: $\phi(t) = e^{\frac{t^2}{2}}$.
This implies $m_{2r} = \dfrac{(2r)!}{r!2^r}$ and $m_{2r-1} = 0$.

3 The Proof (of repeated same prob. n times)

This is my summarization from the paper "The automatic central limit theorems generator (and much more!)" of Dr.Z.

A convenient way to encode this via, *probability generating function*,

$$f(t) := \sum_{r \in R} P(X(s) = r) t^r,$$

This is easily seen to be equal to

$$\sum_{s \in S} p_s t^{X(s)}.$$

Let $f_r(X)$ be the factorial moment

$$f_r(X) := \sum_{s \in S} p_s(X(s) - \mu)^{(r)}$$

where $X^{(r)} = X(X-1)\dots(X-r+1)$.

Also define F_r by $F_r = \frac{d^r f(t)}{dt^r}\Big|_{t=1}$ and $f_r(n) = \frac{d^r f(t)^n}{dt^r}\Big|_{t=1}$.

Note $f_r(1) = F_r$.

3.1 How to Proof

Concept

We'd like to show the moment $E[e^{tX}]$ is the same as the moment of the normal distribution, $e^{\frac{t^2}{2}}$. We need to do so by showing that

$$m_{2r}(n) = m_2(n)^r \cdot \frac{(2r)!}{2^r \cdot r!}\left(1 + \frac{P_1(r)}{n} + \frac{P_2(r)}{n^2} + \cdots\right) \qquad \text{and}$$

$$m_{2r+1}(n) = m_2(n)^{r+\frac{1}{2}}\left(\frac{P_1(r)}{n} + \frac{P_2(r)}{n^2} + \cdots\right).$$

We will do this by first conjecture the formula of $f_r(n)$ by differentiate the *probability generating function*. Second, prove it using the recurrence in step 2. Lastly connect $f_r(n)$ to $m_r(n)$ using Stirling numbers of the second kind in step 3.

Step 1 Calculate F_r from Maclaurin series of $f(1+z)$ around $t = 0$,

$$f(1+z) = \sum_{r=0}^{\infty} \frac{F_r z^r}{r!},$$

or by using the relation $F_r = \frac{d^r f(t)}{dt^r}\Big|_{t=1}$.

Then also conjecture formula of $f_r(n)$,

$$f_r(n) = \frac{d^r f(t)^n}{dt^r}\Big|_{t=1}.$$

Step 2 Prove the conjecture about $f_r(n)$ using the recurrence:

$$f_r(n) = \sum_{j=0}^{r-1}\left[n\binom{r-1}{j} - \binom{r-1}{j+1}\right] \cdot F_{j+1} \cdot f_{r-1-j}(n)$$

where $f_0(n) = 1$ and $f_1(n) = 0$.

This recurrence is obtained by differentiate both sides of

$$f(1+z)^n = \sum_{r=0}^{\infty} \frac{f_r(n) z^r}{r!},$$

then multiply both sides with $f(1+z)$, rearrange the terms and compare the coeff. of z^{r-1}.

Remark We can also another recurrence that is easier to derive:

$$f_r(n) \text{ is defined by } f(1+z)^n = \sum_{r=0}^{\infty} \frac{f_r(n) z^r}{r!}.$$

We use the fact that

$$f(1+z)^{n+1} = f(1+z)^n \cdot f(1+z)$$

that entails:

$$1 + \sum_{r=2}^{\infty} \frac{f_r(n+1)}{r!} z^r = \left(1 + \sum_{r=2}^{\infty} \frac{f_r(n)}{r!} z^r\right)\left(1 + \sum_{r=2}^{\infty} \frac{F_r}{r!} z^r\right).$$

Rearranging, and comparing coefficient of z^r, we have the following *recurrence*

$$f_r(n+1) - f_r(n) = \sum_{s=2}^{r} \binom{r}{s} F_s f_{r-s}(n), \quad n \geq 1.$$

Step 3 Connect $m_r(n)$ with $f_r(n)$ by using the relation

$$m_r(n) = \sum_{k=1}^{r} s_2(r,k) f_k(n)$$

where $s_2(r,k)$ is a Stirling number of second kind defined by the recurrence

$$s_2(r,k) = k s_2(r-1,k) + s_2(r-1,k-1),$$

$s_2(1,k) = 1$ if $k = 1$ and $s_2(1,k) = 0$, otherwise.

Note $X^r = \sum_{k=1}^{r} s_2(r,k) X^{(k)}$.

4 The probability distribution of number of head after tossing a fair coin n times is asymptotically normal

4.1 Binomial distribution with $p = \dfrac{1}{2}$

First moment, $E[X] = \left(\frac{1}{2}\right)^n \sum_x \binom{n}{x} x = \frac{n}{2}$.

Second moment, $E[X^2] = \left(\frac{1}{2}\right)^n \sum_x \binom{n}{x} x^2 = \frac{n(n+1)}{4}$.

Second moment about the mean, $E[(X-\mu)^2] = \left(\frac{1}{2}\right)^n \sum_x \binom{n}{x} (x-\mu)^2 = \frac{n}{4}$.

$m_{2r+1}(n) = 0$, $m_4(n) = \frac{n(3n-2)}{16}$ and ...

The general form of $m_{2r}(n) = c_r n^r + O(n^{r-1})$ where $c = \left[\frac{1}{4}, \frac{3}{16}, \frac{15}{64}, \frac{105}{256}, \frac{945}{1024}, ...\right]$.

This seemingly random looking polynomials actually have pattern.

$$m_{2r}(n) = \left(\frac{n}{4}\right)^r \frac{(2r)!}{2^r \cdot r!} \left(1 - \frac{r(r-1)}{3n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right).$$

But how to prove $m_{2r}(n)$ is true up to certain order rigorously?

4.2 Probability generating function

Let the *probability generating function*, $g(t) = \frac{1+t}{2}$ and the probability generating function about the mean, $f(t) = \frac{1+t}{2} \cdot \frac{1}{\sqrt{t}}$.

4.3 Proof

We only need to show $f_{2r}(n)$ and $f_{2r+1}(n)$ are true up to certain order.

Conjecture: $f_{2r}(n) = \left(\frac{n}{4}\right)^r \frac{(2r)!}{2^r \cdot r!} \left(1 + \frac{r(r-1)(4r-1)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)$.

and $f_{2r+1}(n) = \left(\frac{n}{4}\right)^r \frac{(2r)!}{2^r \cdot r!} \left(-(2r+1)r - \frac{r^2(r-1)(2r+1)(4r+1)}{3n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)$.

We can prove these equations (PLUS the smaller terms) by using the recurrence:

$$f_r(n) = \sum_{j=0}^{r} \left[n\binom{r-1}{j} - \binom{r-1}{j+1} \right] \cdot F_{j+1} \cdot f_{r-1-j}(n),$$

where $F_i = \left[1, 0, \frac{1}{4}, -\frac{3}{4}, \frac{45}{16}, ...\right], i = 0, 1, 2, ...$

Example 1

Say we want to show: $f_{2r}(n) = \left(\frac{n}{4}\right)^r \frac{(2r)!}{2^r \cdot r!}$ + smaller order.

Proof. By induction on r
For $r = 0$, $f_0(n) = 1$.

Induction step assume the statement is true for $r = 0, 1, ..., R-1$.

Then by the recurrence

$$\begin{aligned} f_{2R}(n) &= \underbrace{n \cdot F_1 \cdot f_{2R-1}(n)}_{0} + n(2R-1)F_2 f_{2R-2}(n) + \text{smaller order} \\ &= n(2R-1)\frac{1}{4}\left(\frac{n}{4}\right)^{R-1} \frac{(2R-2)!}{2^{R-1}(R-1)!} + \text{smaller order} \\ &= \left(\frac{n}{4}\right)^R \frac{(2R)!}{2^R R!} + \text{smaller order} \end{aligned}$$

□

Example 2

To show: $f_{2r+1}(n) = \left(\frac{n}{4}\right)^r \frac{(2r)!}{2^r \cdot r!} \left(-\binom{2r+1}{2}\right)$ + smaller order.

Proof.

$$\begin{aligned} f_{2R+1}(n) &= n\binom{2R}{1} \cdot F_2 \cdot f_{2R-1}(n) + n\binom{2R}{2} F_3 f_{2R-2}(n) + \text{smaller order} \\ &= \left(\frac{n}{4}\right)^{R-1} \frac{(2R-2)!}{2^{R-1} \cdot (R-1)!} \left[-\frac{n(2R)}{4}(2R-1)(R-1) - \frac{n(2R)(2R-1)}{2} \cdot \frac{3}{4}\right] + \text{smaller order} \\ &= \left(\frac{n}{4}\right)^{R-1} \frac{(2R-2)!}{2^{R-1} \cdot (R-1)!} \left(\frac{n}{4}\right)(2R)(2R-1)\left[-R + 1 - \frac{3}{2}\right] + \text{smaller order} \\ &= \left(\frac{n}{4}\right)^R \frac{(2R)!}{2^{R-1} \cdot R!} R\left(-\frac{2R+1}{2}\right) + \text{smaller order} \\ &= \left(\frac{n}{4}\right)^R \frac{(2R)!}{2^R \cdot R!}[-(2R+1)R] + \text{smaller order.} \end{aligned}$$

□

Example 3

To show: $f_{2r}(n) = \left(\frac{n}{4}\right)^r \frac{(2r)!}{2^r \cdot r!}\left(1 + \frac{r(r-1)(4r-1)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right).$

Proof. Use

$$\begin{aligned} f_{2R}(n) =& n\binom{2R-1}{1}F_2 \cdot f_{2R-2}(n) - \binom{2R-1}{2}F_2 \cdot f_{2R-2}(n) \\ &+ n\binom{2R-1}{2}F_3 f_{2R-3}(n) + n\binom{2R-1}{3}F_4 f_{2R-4}(n) \\ &+ \text{smaller order.} \end{aligned}$$

□

5 The probability distribution of # of Inversion is asymptotically normal

5.1 Number of inversion on permutation of length n

Another important discrete distribution function is the *Mahonian* distribution, defined on the set of *permutations* on n objects, describing the random variable "number of inversions" ($inv(\pi)$ is the number of pair $(i, j), 1 \leq i < j \leq n$ such that $\pi_i > \pi_j$).

First moment, $E[X] = \dfrac{n(n-1)}{4}$.
Second moment about the mean, $E[(X-\mu)^2] = \dfrac{n(n-1)(2n+5)}{72}$.

To see how we find these moments: $X_n = Y_1 + \cdots + Y_n$, where $Y_1, \ldots, Y_n$ are independent random variables and Y_j is uniformly distributed on $\{0, \ldots, j-1\}$.

5.2 Proof

Probability generating function

$$F_n(q) = \frac{1}{n!}\prod_{i=1}^{n}\frac{1-q^i}{1-q}.$$

$$G_n(q) = \frac{F_n(q)}{q^{\frac{n(n-1)}{4}}}.$$

$B_r(n)$, binomial moment $(= \mathrm{E}\left[\binom{M_n}{r}\right])$, can be calculated from

1. $G_n(1+z) = \sum_{r=0}^{\infty} B_r(n) z^r$ or
2. Let $P_n(q) := \dfrac{G_n(q)}{G_{n-1}(q)} \to P_n(q) = \dfrac{1}{n} \cdot \dfrac{1-q^n}{1-q} \cdot q^{\frac{1-n}{2}}$

Then $B_r(n)$ can be calculated from the recurrence
(from $G_n(1+z) = P_n(1+z)G_{n-1}(1+z)$ and comparing coef. of z^r.)

$$B_r(n) - B_r(n-1) = \sum_{s=2}^{r} B_{r-s}(n-1)p_s(n)$$

where $p_s(n)$ is from $P_n(1+z) = \sum_{i=0}^{\infty} p_i(n)z^i$. Note $p_0(n) = 1$ and $p_1(n) = 0$.

Then moment about mean $M_r(n) = \sum_{k=1}^{r} s_2(r,k) \cdot B_k(n) \cdot k!$.
(similar relation to the last section.)

Theorem 1.

$$\frac{M_{2r}(n)}{M_2(n)^r} = \frac{(2r)!}{2^r r!}\left(1 - \frac{9r(r-1)}{25} \cdot \frac{1}{n}\right) + \mathcal{O}\left(\frac{1}{n^2}\right).$$

Proof. Notice that here we will follow the same procedure. We first conjecture $B_r(n)$. Then prove it using the recurrence. And finally we convert $B_r(n)$ to $M_r(n)$ to prove the theorem.

Let just try to show this only for the leading term.
Note that
1. $M_2(n) = \frac{n^3}{36} + \mathcal{O}(n^2)$.
2. By induction, we also know $\mathrm{Deg}(B_{2r}) > \mathrm{Deg}(B_{2r-1})$.

We use Maple to conjecture the formula of $B_{2r}(n)$:

$$B_{2r}(n) = \left(\frac{n^3}{72}\right)^r \cdot \frac{1}{r!} + \mathcal{O}(n^{3r-1}).$$

Note $p_0(n) = 1, p_1(n) = 0$ and $p_2(n) = \frac{n^2-1}{24}$.

We prove by induction on r.

Base case: $B_2(n) = \frac{n^3}{72} + \mathcal{O}(n^2)$.
Induction step: Assume true up to $r-1$

$$\begin{aligned} B_{2r}(n) - B_{2r}(n-1) &= \frac{n^2}{24} \cdot \left(\frac{n^3}{72}\right)^{r-1} \cdot \frac{1}{(r-1)!} + \mathcal{O}(n^{3r-2}) \\ &= \frac{n^{3r-1}}{24 \times 72^{(r-1)}} \cdot \frac{1}{(r-1)!} + \mathcal{O}(n^{3r-2}). \end{aligned}$$

This implies

$$B_{2r}(n) = \frac{n^{3r}}{72 \times 72^{r-1}} \cdot \frac{1}{r(r-1)!} + \mathcal{O}(n^{3r-1})$$
$$= \frac{n^{3r}}{72^r} \cdot \frac{1}{r!} + \mathcal{O}(n^{3r-1}).$$

□

6 Conclusion and Future Work

6.1 Conclusion

Using *symbolic-crunching* the computer can derive deep theorems. The passage from the discrete to the continuous becomes much more concrete and down-to-earth.

6.2 Future Work

Is the distribution of everything is asymptotically normal?

		Proof	section
1.	Binomial distribution	Here	4
2.	Number of inv. on $n!$	Here	5
3.	Number of des. on $n!$	Here	??
4.	Number of Maj132 on $n!$	Here (not done)	??
5.	Mahonian prob. dist. on words	Dr.Z.	–
6.	Inv + Maj join prob. dist.	Dr.Z.	–
7.	Number of pattern avoid. 132 on $n!$	No proof	??

The degree of moment (about mean)

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14
binomial	0	1	0	2	0	3	0	4	0	5	0	6	0	7
inversion	0	3	0	6	0	9	0	12	0	15				
descendent	0	1	0	2	0	3	0	4	0	5	0	6	0	7
Maj132	0	3	4	6	7	9	10	12	13					
pattern avoid. 132	0	5	7	10										

Other possible objects:
Ramsey number, Schur number, Van der Waerden number, Boolean function and Maj+des.