Moment Calculus On Ramsey Graph

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Ramsey theory is a fascinating but extremely difficult subject started by British mathematician Frank Ramsey in the early 1900. But it is Paul Erdös who popularize the field. Although he passed away in 1996, this subject is still alive and gives rise to many interesting research projects.

We will restrict the talk only on Ramsey graph (one of the Super-Six theorems, [4]) today.

1 Introduction to Ramsey Number

$R(k, l)$ is the smallest number of vertices of complete graph which each edge colored either red or blue such that no matter how the edges are colored, it must contain either (monochromatic) red K_k or blue K_l.

Example: $R(3, 3) = 6$.

$$R(4, 4) = 18, \quad 43 \leq R(5, 5) \leq 49, \quad 102 \leq R(6, 6) \leq 165.$$

We see the exact number of $R(k, k)$ is very hard to determine because of the gigantic possibilities of edge-colorings, $2^{\binom{n}{2}}$ ways to color edges.

The question about the asymptotic behavior of $R(k, k)$ is a famous open problem in combinatorics. There is even a monetary prize, \$250, for the answer to the problem. Paul Erdös used the first moment $E[X]$ to obtain a lower bound of $\lim_{k \to \infty} R(k, k)$.

Theorem 1.

$$\sqrt{2}^k \leq R(k, k) \leq 4^k, \quad k \geq 3.$$

Proof. For the upper bound:

Claim: $R(m, n) \leq \binom{m+n}{n}$.

(Once this is shown we will have that

$$R(k, k) \leq \binom{2k}{k} = \frac{(2k)!}{k!k!} \approx \frac{2^{2k} k^{2k}}{k^k \cdot k^k} = 4^k.)$$

We can use the induction on $m+n$ to show the claim. We see that

$$R(m,n) \le R(m-1,n) + R(m,n-1).$$

Then it follows that

$$R(m,n) \le R(m-1,n) + R(m,n-1) \le \binom{m+n-1}{n} + \binom{m+n-1}{n-1} = \binom{m+n}{n}.$$

For the lower bound (Erdös, 1947):

We use the property that

$$1 - E[X] \ge 0 \to P(X=0) > 0.$$

Here we let the random variable X be the number of mono-chromatic subgraph of size k in the complete graph of n vertices. We want to find n such that $E[X] < 1$ then we have $R(k,k) > n$ since there is some random variable (random edge-coloring graph) that does not contain monochromatic red K_k or monochromatic blue K_k.

Here

$$E[X] = \frac{(n)_k}{k!} \cdot \frac{2}{2^{\binom{k}{2}}} \approx \frac{n^k}{k!} \cdot \frac{2}{2^{\binom{k}{2}}}.$$

If $n \le \sqrt{2}^k$ then

$$E[X] \le \frac{2^{k^2/2} \cdot 2}{k! \cdot 2^{k^2/2 - k/2}} = \frac{2^{k/2+1}}{k!} < 1 \quad \text{for } k \ge 3.$$

$\square$

Remark that the precise bound from this idea is

$$R(k,k) \ge \frac{1}{\sqrt{2}e} k 2^{k/2} (1 + o(1)), \quad k \to \infty.$$

The bound can be improved by using more sophisticated technique called *Lovasz local lemma*, see [1],

$$R(k,k) \ge \frac{\sqrt{2}}{e} k 2^{k/2} (1 + o(1)), \quad k \to \infty.$$

Prize Money Problems (Ron Graham)

1. (\$100) Does $\lim_{k\to\infty} R(k,k)^{\frac{1}{k}}$ exist?
2. (\$250) If the limit exists, what is it?

The idea that Erdös used for the lower bound can be extended by the method called *moment calculus*.

2 Moment Calculus of Ramsey Graphs

Let S be k-subsets of $\{1, 2, \ldots, n\}$.
Let X_S be an indicator variable.

$$X_S = \begin{cases} 1 & \text{if subgraph of } K_n \text{ induced by } S \text{ is monochromatic} \\ 0 & \textit{otherwise.} \end{cases}$$

$X = \sum_S X_S$ (recall X is the number of monochromatic K_k in this specific graph.)

First moment:

$$E[X_S] = \frac{2}{2^{\binom{k}{2}}},$$
$$E[X] = \frac{2}{2^{\binom{k}{2}}} \cdot \binom{n}{k}.$$

Second moment:

$$E[X^2] = E\left[(\sum_{S_1} X_{S_1})(\sum_{S_2} X_{S_2}) \right] = \sum_{[S_1, S_2]} E[X_{S_1} X_{S_2}].$$

We need to look at how S_1 and S_2 interact with each other.

Example: For $k = 3$,

$$E[X^2] = \frac{2}{2^3} \cdot \frac{2}{2^3} \cdot \frac{(n)_6}{3!3!} + \frac{2}{2^3} \cdot \frac{2}{2^3} \cdot \frac{(n)_5}{2!2!1!} + \frac{2}{2^5} \cdot \frac{(n)_4}{2!1!1!} + \frac{2}{2^3} \cdot \frac{(n)_3}{3!}.$$

For $k = 4$,

$$E[X^2] = \frac{2}{2^6} \cdot \frac{2}{2^6} \cdot \frac{(n)_8}{4!4!} + \frac{2}{2^6} \cdot \frac{2}{2^6} \cdot \frac{(n)_7}{3!3!1!} + \frac{2}{2^{11}} \cdot \frac{(n)_6}{2!2!2!} + \frac{2}{2^9} \cdot \frac{(n)_5}{3!1!1!} + \frac{2}{2^6} \cdot \frac{(n)_4}{4!}.$$

In fact, we can write the formula of second moment for general k as a sum (not the closed form though).

$$E[X^2] = \frac{2}{2^{\binom{k}{2}}} \cdot \frac{2}{2^{\binom{k}{2}}} \cdot \frac{(n)_{2k}}{k!k!} + \frac{2}{2^{\binom{k}{2}}} \cdot \frac{2}{2^{\binom{k}{2}}} \cdot \frac{(n)_{2k-1}}{1!(k-1)!(k-1)!} + \sum_{i=2}^{k} \frac{2}{2^{2\binom{k}{2} - \binom{i}{2}}} \cdot \frac{(n)_{2k-i}}{i!(k-i)!(k-i)!}.$$

For higher moment with fixed k, we need computer to do the job for us. There are too many ways the objects can interact with each other. The program I wrote can calculate up to the fifth moment for some small k.

One nice thing about this calculation is that you can check the correctness of your formula by comparing it with the moment from different calculation (small k).

$$E[X^r] = \sum_{i=0}^{\infty} i^r P[X = i].$$

3 Normal Distribution of # of Monochromatic Complete Subgraphs

The following results come from Maple program.

Theorem 2. *The leading term of $E[(X-\mu)^2]$ is*

$$\frac{1}{2} \cdot \frac{1}{(k-3)!^2} \cdot \frac{n^{2k-3}}{2^{2\binom{k}{2}-2}}.$$

The leading term of $E[(X-\mu)^3]$ is

$$\frac{1}{(k-3)!^3} \cdot \frac{n^{3k-5}}{2^{3\binom{k}{2}-3}}.$$

The leading term of $E[(X-\mu)^4]$ is

$$\frac{3}{4} \cdot \frac{1}{(k-3)!^4} \cdot \frac{n^{4k-6}}{2^{4\binom{k}{2}-4}}.$$

The leading term of $E[(X-\mu)^5]$ is

$$5 \cdot \frac{1}{(k-3)!^5} \cdot \frac{n^{5k-8}}{2^{5\binom{k}{2}-5}}.$$

With these results, we've already seen an asymptotic normality of X when $n \gg k$.

Corollary 3. *As $k \to \infty$ and $n \geq \frac{\sqrt{2}k}{e} 2^{\frac{k}{2}}(1+o(1))$, the random variable X is normally distributed.*

(The condition of n is needed to make the leading term significance.)

Proof. We show that the standardized moments $c_m := \frac{E[(X-\mu)^m]}{Var^{\frac{m}{2}}}$ agree with the coefficient of the moment generating function of standard normal distribution $e^{\frac{t^2}{2}}$ i.e. $0, 1, 0, 3, 0, 15, 0, 105, 0, 945, \ldots$.

From theorem 2, we see

$$\begin{aligned}
c_1 &= \frac{0}{\sqrt{Var}} = 0, \\
c_2 &= 1, \\
c_3 &= \frac{n^{3k-5}}{n^{3k-9/2}/(2\sqrt{2})} = \frac{2\sqrt{2}}{\sqrt{n}} \to 0 \quad \text{as } n \to \infty, \\
c_4 &= \frac{3}{4} \cdot 2^2 = 3. \\
c_5 &= \frac{5n^{5k-8}}{n^{5k-15/2}/(4\sqrt{2})} = \frac{20\sqrt{2}}{\sqrt{n}} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

The other values will conform as well. □

This is a complement of the previous known result:

In [2], it was shown that X_k is asymptotically Poisson as $k \to \infty$ with condition $n \leq \frac{\sqrt{2}}{e} k 2^{k/2}(1+o(1))$. That is we have

$$P(X_k = j) \approx \frac{\lambda^j e^{-\lambda}}{j!}, \quad \text{where } \lambda = \frac{\binom{n}{k}}{2^{\binom{k}{2}-1}}.$$

Remark. It is quite evidence that, asymptotically ($k \to \infty$), X does not have a Poisson distribution through out. The leading term of $Var(X)$ is $\frac{1}{2(k-3)!^2} \cdot \frac{n^{2k-3}}{2^{2\binom{k}{2}-2}}$ and $\frac{Var(X)}{E[X]} \sim \frac{k(k-1)(k-2)}{(k-3)!} \cdot \frac{n^{k-3}}{2^{\binom{k}{2}}}$.

Hence, for some "**big** n", and $k \geq 4$, $Var(X) \gg E[X]$.

In conclusion, we can (supposedly) show, asymptotically, that for big n, X is normally distributed.

3.1 Almost Surely Property

We will apply Chebyshev's theorem for the almost surely property of $R(k,k)$.

Theorem 4 (Chebyshev's theorem)**.** *Let X be a discrete random variable ≥ 0,*

$$P(X=0) \leq \frac{Var(X)}{E[X]^2}.$$

Theorem 5. *For $n \geq \frac{\sqrt{2}}{e} k 2^{k/2}(1+o(1))$, as $k \to \infty, P(X=0) \to 0$, almost surely.*

Proof. From theorem 2, we have

$$\frac{Var(X)}{E[X]^2} \sim \frac{k^6}{n^3}.$$

We then see that $\frac{Var(X)}{E[X]^2} \to 0$, then the result follows from Chebyshev's theorem. $\square$

4 Delaporte Distribution

In [3], the authors found the best fit for the distribution of X to be Delaporte. We will discuss this distribution in this section.

Definition (Delaporte distribution)**.**

Let $mgf(X) = E[e^{tX}]$. We define Delaporte distribution by moment generating function

$$mgf(D) = \frac{e^{\lambda(e^t-1)}}{(1-\beta(e^t-1))^\alpha}.$$

The motivation behind this is that D is a convolution of a Negative binomial random variable with success probability $\frac{\beta}{1+\beta}$ and mean $\alpha\beta$ and a Poisson random variable with mean λ.

Corollary 6. *For Delaporte distribution,*

$$P(D=j) = \sum_{i=0}^{j} \frac{\Gamma(\alpha+i)}{\Gamma(\alpha)i!} \left(\frac{\beta}{1+\beta}\right)^i \left(\frac{1}{1+\beta}\right)^\alpha \frac{\lambda^{j-i}e^{-\lambda}}{(j-i)!}.$$

It also follows that

$$\begin{aligned}
\mu = E[X] &= \lambda + \alpha\beta, \\
Var(X) = E[(x-\mu)^2] &= \lambda + \alpha\beta(1+\beta), \\
E[(X-\mu)^3] &= \lambda + \alpha\beta(1+3\beta+2\beta^2), \\
E[(X-\mu)^4] &= 3\lambda^2 + \lambda + \alpha\beta(1+\beta)(3\alpha\beta^2+3\alpha\beta+6\beta^2+6\beta+6\lambda+1), \\
&\cdots
\end{aligned}$$

Proof. The probability mass function can be calculated from $mgf(D)$.

The moment generating function for Poisson:

$$\begin{aligned}
mgf(P) &= \sum_i \frac{e^{-\lambda}\lambda^i}{i!}e^{ti} \\
&= e^{-\lambda}\sum_i \frac{(\lambda e^t)^i}{i!} \\
&= e^{\lambda(e^t-1)}.
\end{aligned}$$

The moment generating function for Negative Binomial:

$$\begin{aligned}
mgf(NB) &= \sum_i \frac{\Gamma(\alpha+i)}{\Gamma(\alpha)\cdot i!}p^i(1-p)^\alpha e^{ti} \quad ,\text{where } p = \frac{\beta}{1+\beta} \\
&= \frac{(1-p)^\alpha}{[1-pe^t]^\alpha} \\
&= \frac{1}{[1+\beta-\beta e^t]^\alpha} \\
&= \frac{1}{[1-\beta(e^t-1)]^\alpha}
\end{aligned}$$

Hence the $mgf(D)$ is the product of $mgf(P)$ and $mgf(NB)$. The probability mass function of D is the convolution of $P(P)$ and $P(NB)$.

The calculation of the moments in the second part is from direct calculation of the generating function. □

5 Asymptotic/Non-asymptotic fit with Delaporte distribution (?)

We will discuss the Delaporte distribution as the fit of X in three scenarios:

1. $k \to \infty$ for "small n".
2. $k \to \infty$ for "big n".
3. small k.

5.1 Delaporte Fit as $k \to \infty$ for "small n"

Theorem 3 in [3] is important but was done in a confusing way. I will write my own version as follows:

Theorem 7. *If $D \sim Delaporte(\lambda, \alpha, \beta)$, and $P \sim Poisson(\lambda+\alpha\beta)$, then $mgf(D) \to mgf(P)$ as $k \to \infty$ under the assumption:*

$$\alpha\beta^2 \to 0.$$

Proof. Indeed, to make the asymptotic distribution of $X \sim$ Poisson, under the condition that $n \leq \frac{\sqrt{2}-\epsilon}{e} k 2^{k/2}(1 + o(1))$, as already been proved in [2], we need to match term of each moment. Setting

$$\mu = E[X] = \lambda + \alpha\beta$$

and

$$\mu = Var(X) = \lambda + \alpha\beta + \alpha\beta^2$$

resulting to the condition that $\alpha\beta^2 \to 0$. Under this condition, the other higher moments fit perfectly as well. □

Indeed the set up of α and β below (along with this condition of n) gives $\alpha\beta^2 \to 0$. It looks like things fit together very well.

5.2 Delaporte Fit as $k \to \infty$ for "big n"

But how to set α, β and λ of Delaporte distribution. We assume the opposite that $n \geq \frac{2k}{e} \cdot 2^{\frac{k}{2}}$. This will results that

$$\frac{Var(X)}{E[X]} = \frac{k(k-1)(k-2)}{(k-3)!} \cdot \frac{n^{k-3}}{2^{\binom{k}{2}}} \gg 1 \quad \text{as } k \to \infty.$$

Hence, we can solve these variables by matching corollary 6 and theorem 2:

$$\begin{aligned} \lambda + \alpha\beta &= \frac{1}{k!} \cdot \frac{n^k}{2^{\binom{k}{2}-1}}, \\ \alpha\beta^2 &= \frac{1}{2(k-3)!^2} \frac{n^{2k-3}}{2^{2\binom{k}{2}-2}}, \\ 2\alpha\beta^3 &= \frac{1}{(k-3)!^3} \frac{n^{3k-5}}{2^{3\binom{k}{2}-3}}. \end{aligned}$$

Then we have:

$$\alpha = \frac{n}{2}, \quad \beta = \frac{n^{k-2}}{2^{\binom{k}{2}-1}} \cdot \frac{1}{(k-3)!},$$
$$\lambda = E[X] - \alpha\beta = \frac{n^k}{k! \cdot 2^{\binom{k}{2}-1}} \left[1 - \frac{k(k-1)(k-2)}{2n}\right].$$

Remark. Assume $n \sim \frac{2k}{e} \cdot 2^{\frac{k}{2}}$, we have

$$\alpha = \frac{n}{2} = \frac{k}{e} \cdot 2^{\frac{k}{2}}, \quad \beta = (\frac{k}{e})^{k-2} \cdot \frac{2^{\frac{k}{2}}}{4} \cdot \frac{2}{(k-3)!} = \frac{k}{e} \cdot \frac{2^{\frac{k}{2}}}{2} \cdot \frac{e^3}{\sqrt{2\pi k}} = \frac{e^2}{2\sqrt{2\pi}} \cdot \sqrt{k} \cdot 2^{\frac{k}{2}},$$
$$\lambda = \frac{2}{\sqrt{2\pi k}} 2^{\frac{k}{2}}.$$

We see that, with this assumption, $\alpha \gg \beta \gg \lambda$, as $k \to \infty$.

With this setting of α, β and λ, the leading terms of the moment of Delaporte (corollary 6) approaches the normal distribution as well, i.e.

$$\begin{aligned}
E[(X-\mu)^2] &\sim \alpha\beta^2 \\
E[(X-\mu)^3] &\sim 2\alpha\beta^3 \\
E[(X-\mu)^4] &\sim 3\alpha^2\beta^4 \\
E[(X-\mu)^5] &\sim 20\alpha^2\beta^5 \\
E[(X-\mu)^6] &\sim 15\alpha^3\beta^6 \\
E[(X-\mu)^7] &\sim 210\alpha^3\beta^7 \\
E[(X-\mu)^8] &\sim 105\alpha^4\beta^8 \\
E[(X-\mu)^9] &\sim 2520\alpha^4\beta^9 \\
E[(X-\mu)^{10}] &\sim 945\alpha^5\beta^{10} \\
&\cdots
\end{aligned}$$

The coefficient of $(2k+3)^{th}$ moment is $\dfrac{(2k+3)!}{3(k!)2^k}$.

5.3 Delaporte for $R(k,k)$, small k

Aaron successfully fitted the Delaporte (α, β, λ) to the (simulation of) distribution of X for $k = 4, 5$ with various n. However the method of moments to find good fit with α, β, λ does not work out well. I could not solve for the values of these variables like mentioning in Aaron's paper. May be there is no methodology for the fit of this distribution after all.

5.4 Conclusion

The method of moments verify that, asymptotically, Delaporte distribution is a good fit for X for both small n and big n cases.

Appendix: Bonferroni's Inequality

These calculations of Bonferroni helps me understand moment calculus better.

Definition (Moment Generating Function)**.**

$$G_X(z) = \sum_{i=0}^{\infty} P(X = i) z^i.$$

Theorem 8 (Inclusion-Exclusion Principle)**.**

$$P(X = 0) = E\left[\binom{X}{0}\right] - E\left[\binom{X}{1}\right] + E\left[\binom{X}{2}\right] - \cdots .$$

Proof. Consider Taylor series expansion about $x = 1$,

$$f(x) = f(1) + \frac{f'(1)(x-1)}{1!} + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \cdots .$$

Then

$$G_X(0) = G_X(1) - \frac{G'_X(1)}{1!} + \frac{G''_X(1)}{2!} - \frac{G'''_X(1)}{3!} + \cdots .$$

which implies the statement of theorem. □

To compute $P(X = 0)$ from this equation seems out of reach. We might not need exact formula anyway. We may just want to use Bonferroni's inequality to improve the lower bound.

Corollary 9 (Bonferroni's inequality:)**.**

For any odd m,

$$P(X = 0) \geq \sum_{s=0}^{m} (-1)^s E\left[\binom{X}{s}\right], \tag{1}$$

For any even m,

$$P(X = 0) \leq \sum_{s=0}^{m} (-1)^s E\left[\binom{X}{s}\right].$$

Erdös used $m = 1$ to get the lower bound $\lim\limits_{k\to\infty} R(k,k)^{\frac{1}{k}} \geq \sqrt{2}$.

$$1 - E[X] > 0 \quad \to \quad P[X = 0] > 0$$

However with $m = 3$ or $m = 5$ do not improve the lower bound at all.

$$1 - E[X] + E\left[\binom{X}{2}\right] - E\left[\binom{X}{3}\right] > 0 \quad \to \quad P[X = 0] > 0$$

$$1 - E[X] + E\left[\binom{X}{2}\right] - E\left[\binom{X}{3}\right] + E\left[\binom{X}{4}\right] - E\left[\binom{X}{5}\right] > 0 \quad \to \quad P[X = 0] > 0$$

Poisson Paradigm

Recall the probability mass function of Poisson distribution:

$$P\{X = j\} = \frac{\lambda^j e^\lambda}{j!} \qquad \text{for } j \geq 0.$$

Crash course in probability:

Moment generating function:

$$\phi(z) = \sum_{j=0}^{\infty} P(X = j) z^j = \sum_{j=0}^{\infty} \frac{\lambda^j e^\lambda}{j!} z^j = e^{(z-1)\lambda}.$$

and,

$$E[(X)_m] = \sum_{j=0}^{\infty} P(X = j)(j)_m = \frac{d^m \phi(z)}{dz^m}_{z=1} = \lambda^m.$$

We can also verify the inclusion-exclusion principle:

$$\sum_{s=0}^{\infty} (-1)^s E\left[\binom{X}{s}\right] = \sum_{s=0}^{\infty} \frac{(-1)^s \lambda^s}{s!} = e^{-\lambda} = P(X = 0).$$

Exponential Moment Generating Function:

$$M_X(t) = E[e^{tX}] = \sum_{j=0}^{\infty} P(X = j) e^{tj} = \sum_{j=0}^{\infty} \frac{\lambda^j e^\lambda}{j!} e^{tj} = e^{\lambda(e^t - 1)}.$$

and,

$$E[X^m] = \sum_{j=0}^{\infty} P(X = j) j^m = \frac{d^m M_X(t)}{dt^m}_{t=0} = \sum_{k=0}^{m} \left\{ {m \atop k} \right\} \lambda^k.$$

where $\left\{ {m \atop k} \right\}$ is the Stirling number of second kind.

Crash course in Stirling number:

Matrix $s(n, k)$, first kind, and matrix $S(n, k)$, second kind, are inverse of each other.

Stirling numbers of first and second kind are dual pair, i.e.

$$a_n = \sum_{k=0}^{n} s(n, k) b_k \iff b_n = \sum_{k=0}^{n} S(n, k) a_k.$$

Two examples of these important identity are

$$(x)_n = \sum_{k=0}^{n} s(n, k) x^k \iff x^n = \sum_{k=0}^{n} S(n, k)(x)_k,$$

and

$$E[X^m] = \sum_{k=0}^{m} S(m,k)\lambda^k \iff \lambda^m = \sum_{k=0}^{m} s(m,k)E[X^k] = E[(X)_m].$$

As mentioned previously, this Poisson case only valid for "small n". Therefore it does not improve the lower bound of $R(k,k)$.

Paradigm Shift: Delaporte Paradigm

Imagine the size of $n \sim \frac{2k}{e}2^{\frac{k}{2}}$ then

$$\lambda \sim 2^{\frac{k}{2}}, \quad \alpha \sim k2^{\frac{k}{2}}, \quad \beta \sim k2^{\frac{k}{2}}.$$

For each $E[(X)_s]$,
The first term is λ^s.
The second term is $s\alpha\beta\lambda^{s-1}$.
The third term is $\binom{s}{2}\alpha\beta^2(\alpha+1)\lambda^{s-2}$.
The fourth term is $\binom{s}{3}\alpha\beta^3(\alpha+1)(\alpha+2)\lambda^{s-3}$.
The fifth term is $\binom{s}{4}\alpha\beta^4(\alpha+1)(\alpha+2)(\alpha+3)\lambda^{s-4}$.
...

Therefore,

$$\begin{aligned}
P(X=0) &= \sum_{s=0}^{\infty}(-1)^s E\left[\binom{X}{s}\right] \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s(\lambda^s + s\alpha\beta\lambda^{s-1} + \binom{s}{2}\alpha\beta^2(\alpha+1)\lambda^{s-2} + \binom{s}{3}\alpha\beta^3(\alpha+1)(\alpha+2)\lambda^{s-3} + \dots)}{s!} \\
&= (1 - \alpha\beta + \frac{\alpha\beta^2(\alpha+1)}{2!} - \frac{\alpha\beta^3(\alpha+1)(\alpha+2)}{3!} + \frac{\alpha\beta^4(\alpha+1)(\alpha+2)(\alpha+3)}{4!} - \dots) \cdot e^{-\lambda} \\
&= \frac{e^{-\lambda}}{(1+\beta)^{\alpha}} \to 0,
\end{aligned}$$

which is the result we expect.

References

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