Moment Method on Ramsey Numbers

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April 9, 2018

Introduction

$R(k, l)$ is the smallest number of vertices of complete graph which each edge colored either red or blue such that no matter how the edges are colored, it must contain either (monochromatic) red K_k or blue K_l.

Example: $R(3, 3) = 6$.

$$R(4,4) = 18, \quad 43 \le R(5,5) \le 49, \quad 102 \le R(6,6) \le 165.$$

Introduction

Theorem

$$\sqrt{2}^k \leq R(k,k) \leq 4^k, \quad k \geq 3.$$

Introduction

Prize Money Problems (Ron Graham)

1. (\$100) Does $\lim_{k\to\infty} R(k,k)^{\frac{1}{k}}$ exist?
2. (\$250) If the limit exists, what is it?

Some Probability Backgrounds

$$E[X] = \sum_x xp(x),$$

or alternatively

$$E[X] = \int xp(x)\,dx.$$

$$Var(X) = E[(X-\mu)^2] = E[X^2] - E[X]^2.$$

Some Probability Backgrounds

Probability generating function:
If X is a discrete random variable taking values that are non-negative integers, then the probability generating function of X is

$$G_X(z) = \sum_{i=0}^{\infty} P(X = i) z^i.$$

Example: Probability generating function of Poisson distribution:

$$G_X(z) = \sum_{i=0}^{\infty} e^{-\lambda} \frac{\lambda^i}{i!} z^i = e^{\lambda(z-1)}.$$

Some Probability Backgrounds

Moment Generating Function:
If X is a random variable, and t is a real variable (or an abstract symbol), the Moment Generating Function of X, denoted by $M_X(t)$ is defined by

$$M_X(t) = E[e^{tX}].$$

Example: Moment generating function of the standard normal distribution:

$$M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{tx}\, dx = e^{t^2/2}.$$

Some Probability Backgrounds

Theorem (The Central Limit Theorem)

Let $X_1, \ldots, X_n$ be a sequence of independent and identically distributed random variables, each with mean μ and variance σ^2, then the distribution

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal distribution as n goes to ∞.

Some Probability Backgrounds

Theorem (Chebyshev's inequality)

If X is random variable with mean μ and variance σ^2, then for any value $k > 0$,

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}.$$

Moment Calculus of Ramsey Graphs

Let S be k-subsets of $\{1, 2, \ldots, n\}$.
Let X_S be an indicator variable.

$$X_S = \begin{cases} 1 & \text{if subgraph of } K_n \text{ induced by } S \text{ is monochromatic} \\ 0 & \textit{otherwise}. \end{cases}$$

$X = \sum_S X_S$ (recall X is the number of monochromatic K_k in this specific graph.)

Moment Calculus of Ramsey Graphs

First moment:

$$E[X_S] = \frac{2}{2^{\binom{k}{2}}},$$
$$E[X] = \frac{2}{2^{\binom{k}{2}}} \cdot \binom{n}{k}.$$

Second moment:

$$E[X^2] = E\left[(\sum_{S_1} X_{S_1})(\sum_{S_2} X_{S_2}) \right] = \sum_{[S_1,S_2]} E[X_{S_1} X_{S_2}].$$

Moment Calculus of Ramsey Graphs

Theorem (Moment about the Mean)

The leading term of $E[(X-\mu)^2]$ *is* $\frac{1}{2} \cdot \frac{1}{(k-3)!^2} \cdot \frac{n^{2k-3}}{2^{2\binom{k}{2}-2}}$.

The leading term of $E[(X-\mu)^3]$ *is* $\frac{1}{(k-3)!^3} \cdot \frac{n^{3k-5}}{2^{3\binom{k}{2}-3}}$.

The leading term of $E[(X-\mu)^4]$ *is* $\frac{3}{4} \cdot \frac{1}{(k-3)!^4} \cdot \frac{n^{4k-6}}{2^{4\binom{k}{2}-4}}$.

The leading term of $E[(X-\mu)^5]$ *is* $5 \cdot \frac{1}{(k-3)!^5} \cdot \frac{n^{5k-8}}{2^{5\binom{k}{2}-5}}$.

Moment Calculus of Ramsey Graphs

Corollary

As $k \to \infty$ and $n \geq \frac{\sqrt{2}k}{e} 2^{\frac{k}{2}}(1 + o(1))$, the random variable X is normally distributed.

In another direction:
In [2], it was shown that X_k is asymptotically Poisson as $k \to \infty$ with condition $n \leq \frac{\sqrt{2}}{e} k 2^{k/2}(1 + o(1))$. That is we have

$$P(X_k = j) \approx \frac{\lambda^j e^{-\lambda}}{j!}, \quad \text{where } \lambda = \frac{\binom{n}{k}}{2^{\binom{k}{2}-1}}.$$

Delaporte Distribution

In [3], the authors found the best fit for the distribution of X to be Delaporte. We will discuss this distribution in this section.

Definition (Delaporte distribution)

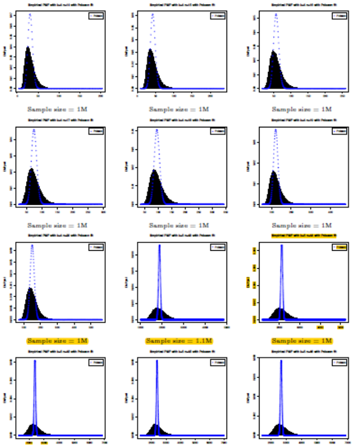
Let $mgf(X) = E[e^{tX}]$. We define Delaporte distribution by moment generating function

$$mgf(D) = \frac{e^{\lambda(e^t-1)}}{(1-\beta(e^t-1))^\alpha}.$$

The motivation behind this is that D is a convolution of a Negative binomial random variable with success probability $\frac{\beta}{1+\beta}$ and mean $\alpha\beta$ and a Poisson random variable with mean λ.

Delaporte Distribution

Best Fit with Poisson distribution



The distribution of K_4, K_5 with various n.

Delaporte Distribution

Best Fit with Delaporte distribution

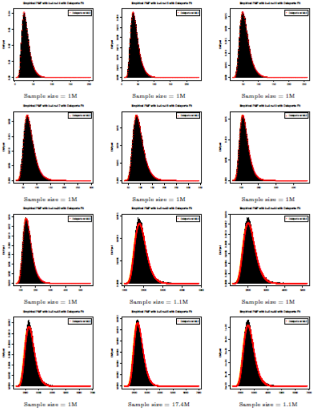


Figure 3: Empirical pmfs for various scenarios with Delaporte Overlay

The distribution of K_4, K_5 with various n.

Delaporte Distribution

Corollary

For Delaporte distribution,

$$P(D=j)=\sum_{i=0}^{j}\frac{\Gamma(\alpha+i)}{\Gamma(\alpha)i!}\left(\frac{\beta}{1+\beta}\right)^{i}\left(\frac{1}{1+\beta}\right)^{\alpha}\frac{\lambda^{j-i}e^{-\lambda}}{(j-i)!}.$$

It also follows that

$$\begin{aligned}
\mu = E[X] &= \lambda+\alpha\beta,\\
Var(X) = E[(x-\mu)^2] &= \lambda+\alpha\beta(1+\beta),\\
E[(X-\mu)^3] &= \lambda+\alpha\beta(1+3\beta+2\beta^2),\\
E[(X-\mu)^4] &= 3\lambda^2+\lambda+\alpha\beta(1+\beta)(3\alpha\beta^2+3\alpha\beta+6\beta^2+6\beta+6\lambda+1),\\
&\cdots
\end{aligned}$$

Delaporte Distribution

Asymptotic/Non-asymptotic fit with Delaporte distribution (?)
We will discuss the Delaporte distribution as the fit of X in three scenarios:

1. $k \to \infty$ for "small n".
2. $k \to \infty$ for "big n".
3. small k.

Delaporte Distribution

Conclusion:
The method of moments verify that, asymptotically, Delaporte distribution is a good fit for X for both small n and big n cases.

References

Aaron, N and Spencer, J, *The Probabilistic Method*, John Wiley and Sons, Inc., New York.

A. Godbole, D. Skipper, and R. Sunley, *The asymptotic lower bound of diagonal Ramsey numbers: a closer look*, Disc. Prob. Algorithms 72 (1995), 81-94.

A. Robertson, W. Cipolli, and M. Dascalu, *On the distribution of monchromatic complete subgraphs and arithmetic progressions*, submitted.

Graham, Ronald L., Rothschild, Bruce L., Spencer, Joel H. and Solymosi, Jozsef, *Ramsey Theory* (3rd ed.) , New York: John Wiley and Sons (2015).

Doron Zeilberger, *Symbolic Moment Calculus II. Why is Ramsey Theory Sooooo Eeeenormoulsy Hard?*, INTEGERS, 7(2)(2007), A34.