SOLUTION 1

1. Solution

Section 1.3 problem 2, page 27

First: Conjecture: $\sum_{k=1}^{n} 2k = n(n+1)$.

Second: We prove by induction:

Base case when k = 1:
Left hand side $\sum_{k=1}^{1} 2k = 2$.
Right hand side 1(1+1) = 2.

Induction step assume the statement is true for k where $1 \leq k < n$.

$$\begin{aligned}
\sum_{k=1}^{n} 2k &= 2 + 4 + 6 + ... + 2(n-1) + 2n \\
&= (n-1)((n-1)+1) + 2n \quad \text{by induction hypothesis} \\
&= n^2 - n + 2n \\
&= n^2 + n \\
&= n(n+1) \qquad \square
\end{aligned}$$

Section 1.3 problem 12, page 27

To show $\sum_{j=1}^{n} j \cdot j! = (n+1)! - 1$.

We again prove by induction:

Base case when j = 1:
Left hand side $\sum_{j=1}^{1} j \cdot j! = 1$.
Right hand side (1+1)!-1 = 2!-1=1.

Induction step assume the statement is true for k where $1 \leq k < n$.

$$\begin{aligned}
\sum_{j=1}^{n} j \cdot j! &= 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + ... + (n-1) \cdot (n-1)! + n \cdot n! \\
&= ((n-1)+1)! - 1 + n \cdot n! \quad \text{by induction hypothesis} \\
&= n! + n \cdot n! - 1 \\
&= (n+1)n! - 1 \\
&= (n+1)! - 1 \qquad \square
\end{aligned}$$

Section 1.4 problem 14, page 34

To show $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ for $n \geq 1$.

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f_n here is Fibonacci sequence. (assume here $f_0 = 0$.)

We prove by induction:

Base case when k = 1:
Left hand side $f_2 f_0 - f_1^2 = 0 - (1)^2 = -1$.
Right hand side $(-1)^1 = -1$.

Induction step assume the statement is true for k where $1 \leq k < n$.
(We need to be a little skillful at shuffling the symbols here).

$$\begin{aligned}
f_{n+1}f_{n-1} - f_n^2 &= (f_n + f_{n-1})f_{n-1} - f_n^2 \\
&= f_{n-1}^2 + f_n f_{n-1} - f_n^2 \\
&= f_{n-1}^2 + f_n(f_{n-1} - f_n) \\
&= f_{n-1}^2 + f_n(-f_{n-2}) \\
&= -(f_n f_{n-2} - f_{n-1}^2) \\
&= -(-1)^{n-1} \quad \text{by induction hypothesis} \\
&= (-1)^n \qquad \Box
\end{aligned}$$

Section 1.4 problem 14, page 39

The two rectangles are composed of the same smaller pieces but have different areas. The catch is the picture on the right is misleading. The slope of the rectangle and the triangle are actually not the same. ($\frac{2}{5}$ against $\frac{3}{8}$).

Section 1.5 problem 36, page 41

To show f_{3k} is even, f_{3k+1} is odd and f_{3k+2} is odd.
(The problem did not state this way, but it is equivalent and easier to work with this way.)

We again use induction. You are doing fine in the homework. Even you did not write the induction out formally but you meant it.

Base case when k = 0:
$f_0 = 0$ is even, $f_1 = 1$ is odd and $f_2 = 1$ is odd.

Induction step assume the statement is true for k where $0 \leq k < n$.

$f_{3n} = f_{3n-1} + f_{3n-2}$ = odd+odd (by induction hypothesis) =even.
$f_{3n+1} = f_{3n} + f_{3n-1}$ = even+odd (by the line above and induction hypothesis) = odd.
$f_{3n+2} = f_{3n} + f_{3n+1}$ = even+odd (by the two lines above) = odd. $\Box$

Section 3.1 problem 6, page 74

To show $n^3 + 1$ is not prime for all integer $n \neq 1$.

Case 1) $n \leq 0$: We have $n^3 + 1 \leq 1$. So $n^3 + 1$ is not prime. (Remember: the smallest prime is 2).

Case 2) $n \geq 2$: We have $n^3 + 1 = (n+1)(n^2 - n + 1)$. Since both $n + 1 \geq 3$ and $n^2 - n + 1 \geq 3$, so $n^3 + 1$ is composite.

$\square$