SOLUTION 2

1. Solution

Problem 1

We translate the given recurrence relation to the characteristic equation.

Characteristic equation:

$x^3 - 3x^2 - 4x + 12 = 0$

We then find the roots of this equation. The good choice to guess the roots is to use the factor of $\frac{a}{b}$ where a is the constant term and b is the leading coefficient. In this case, we use the factors of 12 which are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$ and ± 12.

We quickly find out that 2 is the root of the equation. We use it to factor the given polynomial.

$x^3 - 3x^2 - 4x + 12 = (x - 2)(x^2 - x - 6) = (x - 2)(x - 3)(x + 2)$

Therefore the roots of the equation are 2,-2 and 3.

Hence, the solution is in the form

$a_n = c_1 2^n + c_2 3^n + c_3 (-1)^n$

We then use the given initial conditions to solve for the constants c_1, c_2 and c_3. The solution is

$a_n = 2^n + 3^n$. □

Problem 2

Date: Tuesday, September 23, 2008.

Type the following commands in the Maple program.

a) $rsolve(\{L(n) - L(n-1) - L(n-2) = 0, L(1) = 1, L(2) = 3\}, L);$

b) $L := n- > ((1 + sqrt(5))/2)^n + ((1 - sqrt(5))/2)^n;$

$simplify(L(n)^2 - L(n-1) * L(n+1) - 5 * (-1)^n);$

c) $f := n- > 1/sqrt(5) * (((1 + sqrt(5))/2)^n - ((1 - sqrt(5))/2)^n);$

$simplify(f(2*n) - f(n) * L(n));$

Problem 3

First we use the Euclidean algorithm to find the $gcd(95, 25)$.

95 = 3(25)+20
25 = 1(20)+5
20 = 4(5)

Hence $gcd(95, 25) = 5$.

Second, we write 5 as a linear combination of 95 and 25 by reversing the Euclidean algorithm.

We substitute 20 = 95-3(25) in 25=1(20)+5.

25 = 1(95-3(25))+5.

(4)25+(-1)95 = 5.

To get the solution, we multiply 194 both sides of the equation.

(776)25+(-194)=970.

Hence,

$$\begin{aligned}(x, y) &= (776 + \tfrac{95}{5}k, -194 - \tfrac{25}{5}k)\\ &=(776 + 19k, -194 - 5k) \text{ for any integer } k.\end{aligned}$$

Problem 4

The problem can be translated into the form of diophantine equation:

$10x + 25y = 500.$

By the same method as the previous problem, we obtain

$(-200)10 + (100)25 = 50.$

Hence $(x, y) = (-200 + 5k, 100 - 2k)$ for any integer k.

This problem we want to solution where x and y to be both positive.

So $40 \le k \le 50$.

Therefore, the number of ways to make change is 11 ways.

Section 2.3 problem 6, page 65

I am expected you to do this problem by using definition.

Let m be a positive real number.

To show: $\sum_{j=1}^{n} j^m$ is $O(n^{m+1})$.

Since $\sum_{j=1}^{n} j^m \le \sum_{j=1}^{n} n^m = n^{m+1}$.

Hence, by definition we have $\sum_{j=1}^{n} j^m$ is $O(n^{m+1})$.

Section 2.3 problem 9, page 65

Assume f is $O(g)$.

To show: f^k is $O(g^k)$.

By definition there is a positive integer C such that $f(n) \le Cg(n)$ for sufficient large n.

It follows that $f(n)^k \le C^k g(n)^k$ for sufficient large n.

We can rewritten the above equation as

$f(n)^k \le Lg(n)^k$ for sufficient large n where $L = C^k$.

By definition f^k is $O(g^k)$ $\square$.

Section 3.5 problem 44, page 119

a) To show: $\sqrt[3]{5}$ is irrational.

Proof by contradiction: assume $\sqrt[3]{5} = \frac{a}{b}$ for some positive integer a, b where $gcd(a, b) = 1$.

Hence $5 = (\frac{a}{b})^3$ and then $5b^3 = a^3$.

It follows that $5|a^3$ and $5|a$.

We can write a in the form $a = 5c$ for some positive integer c.

We then have $b^3 = 25c^2$. By similar argument $5|b$.

But this gives contradiction since $5|a$ and $5|b$ but we first assume $gcd(a, b) = 1$.

b) Since $\sqrt[3]{5}$ is the root of $x^3 - 5 = 0$ and it is not an integer. By theorem $\sqrt[3]{5}$ is an irrational number. $\square$

Section 3.5 problem 46, page 119

To show: $log_2 3$ is irrational.

Proof by contradiction: assume $log_2 3 = \frac{a}{b}$ for some positive integer a, b where $gcd(a, b) = 1$.

Hence $2^b = 3^a$.

It follows that $2|3^a$ and $2|3$.
This gives us a contradiction. $\square$