SOLUTION 7

1. Solution

Problem 1

From the program on the web site.

c) There is no $n \leq 50000$ that is a strong pseudoprime to both bases 2 and 3.

Problem 2

The motivation comes from playing around with numbers.
We see $\phi(2) = 2 - 1 = 2$, $\phi(8) = 8 - 4 = 4$, $\phi(32) = 32 - 16 = 16$.

In general $\phi(2^{2n+1}) = 2^{2n+1} - 2^{2n} = 2^{2n}$ for every non-negative integer n.

Problem 3

Write $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, where $p_1, p_2, ..., p_k$ are distinct prime factors of n.

First to show: if n is a perfect square then $\tau(n)$ is odd.

Assume n is a perfect square.
To show $\tau(n)$ is odd.

Since n is a perfect square, all α_i is even.
Therefore $\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_k + 1)$ is odd.

Second to show: if $\tau(n)$ is odd then n is a perfect square.

We prove by contrapositive
Assume n is not a perfect square.
To show $\tau(n)$ is even.

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Since n is not a perfect square, at least one of the α_i is odd (or we say $\alpha_i + 1$ is even).
Therefore $\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1)...(\alpha_k + 1)$ is even.

Problem 4

The trick to do this problem is to realize that
d is a divisor of n if and only if $\frac{n}{d}$ is a divisor of n.

Assume n to be a perfect number.
In another word, assume $\sigma(n) = 2n$.

$$\begin{aligned} \sum_{d|n} \frac{1}{d} &= \frac{1}{n}\sum_{d|n} \frac{n}{d}, \text{ multiply } n \text{ on top and bottom} \\ &= \frac{1}{n}\sum_{d|n} d \\ &= \frac{1}{n}\sigma(n), \text{ by definition} \\ &= \frac{1}{n}2n, \text{ by assumption} \\ &= 2. \ \square \end{aligned}$$

Problem 5

To show: $\tau(n) \leq 2\sqrt{n}$.
Let d be a divisor of n.

Case 1: Consider d where $d \leq \sqrt{n}$.
There are at most $\sqrt{n}$ of d here.

Case 2: Consider d where $d > \sqrt{n}$.
Each d has a one-one correspondence to $d' \leq \sqrt{n}$ where $dd' = n$.
Hence, there are at most $\sqrt{n}$ of d in this case also.

Therefore, the number of divisors of n are at most $\sqrt{n} + \sqrt{n} = 2\sqrt{n}$.

Problem 6

We first mention 2 lemmas.
i) $\sigma(2^n)$ is odd for all non-negative integers n
ii) For a prime $p \geq 3, \sigma(p^\alpha)$ is even if α is odd
and $\sigma(p^\alpha)$ is odd if α is even

From ii), we have that
$\sigma(p^\alpha) \equiv (\alpha + 1) \bmod 2$, for any odd prime p.

Now we can prove the theorem.

Write $n = 2^k p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$

$$\begin{aligned}\sigma(n) &= \sigma(2^k)\sigma(p_1^{\alpha_1})\sigma(p_2^{\alpha_2})...\sigma(p_k^{\alpha_k}) \\ &\equiv 1(\alpha_1+1)(\alpha_2+1)...(\alpha_k+1) \bmod 2 \\ &\equiv \tau(m) \bmod 2, \ m \text{ is the biggest odd factors of } n.\end{aligned}$$

Problem 8 page 245

To show No positive n such that $\phi(n) = 14$.

Consider $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$, where $p_1, p_2, ..., p_k$ are distinct prime factors of n.

$$\begin{aligned}\phi(n) &= \phi(p_1^{\alpha_1})\phi(p_2^{\alpha_2})...\phi(p_k^{\alpha_k}) \\ &= p_1^{\alpha_1-1}(p_1-1)p_2^{\alpha_2-1}(p_2-1)...p_k^{\alpha_k-1}(p_k-1).\end{aligned}$$

If such n exists then $p_i - 1 = 1$ or $p_i - 1 = 2$ or $p_i - 1 = 7$ or $p_i - 1 = 14$ for each i.
This gives the possibilities of p_i to be only 2 or 3.

Therefore n must be in the form $n = 2^{\alpha_1}3^{\alpha_2}$.
However 7 not divide $\phi(n)$.

This shows no such n exists.

Problem 26 page 267

Assume $n = 2^q$ where $2^{q+1} - 1$ is prime.

To show: $\sigma(\sigma(n)) = 2n$.

$\sigma(n) = \sigma(2^q) = \frac{2^{q+1}-1}{2-1} = 2^{q+1} - 1$.

$\sigma(\sigma(n)) = \sigma(2^{q+1} - 1) = (2^{q+1} - 1) + 1$ (since $2^{q+1} - 1$ is prime.)
$= 2^{q+1} = 2n$.

Problem 27 page 267

We first mention 3 facts here.
i) $\sigma(n) = n$ if and only if $n = 1$.
ii) $\sigma(n) \geq n + 1$ for all $n \geq 2$.

iii) For $n \geq 2$, $\sigma(n) = n + 1$ if and only if n is prime.

Assume n to be an even superperfect number.
In another word, assume $\sigma(\sigma(n)) = 2n$ where n is even.

Write $n = 2^k t$ where $k \geq 1$ and t is odd.
$\sigma(n) = \sigma(2^k)\sigma(t) = (2^{k+1} - 1)\sigma(t)$.

$$\begin{aligned} 2n &= \sigma(\sigma(n)) \\ &= \sigma(2^{k+1} - 1)\sigma(\sigma(t)) \\ &\geq (2^{k+1} - 1 + 1)t, \text{ by facts i) and ii)} \\ &= 2n. \end{aligned}$$

This implies $2^{k+1} - 1$ is prime and $t = 1$.
In conclusion $n = 2^k$ and $2^{k+1} - 1$ is prime.