FORMULA SHEET

1. Chapter 2

Newton's Method:

$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad n \geq 1.$

Secant Method:

$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1}-p_{n-2})}{f(p_{n-1})-f(p_{n-2})}, \quad n \geq 2.$

Improved Newton's Method:

$p_n = p_{n-1} - \frac{f(p_{n-1})f'(p_{n-1})}{f'(p_{n-1})^2-f(p_{n-1})f''(p_{n-1})}, \quad n \geq 1.$

Aitken's Δ^2 Method:

$\hat{p_n} = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$, given sequence p_n.

Steffensen's Method:

$p_0^{(n)} = p_0^{(n-1)} - \frac{(p_1^{(n-1)}-p_0^{(n-1)})^2}{p_2^{(n-1)}-2p_1^{(n-1)}+p_0^{(n-1)}}$, given $p_0 \quad (= p_0^{(0)})$.

where $p_1^{(n-1)} = g(p_0^{(n-1)})$ and $p_2^{(n-1)} = g(p_1^{(n-1)})$.

2. Chapter 3

Lagrange Interpolating polynomial:

Given $(x_0, y_0), (x_1, y_1), ..., (x_k, y_k)$.

$L_i(x) = \frac{(x-x_0)(x-x_1)...(x-x_{i-1})(x-x_{i+1})...(x-x_k)}{(x_i-x_0)(x_i-x_1)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_k)}.$

$P(x) = y_0L_0(x) + y_1L_1(x) + ... + y_kL_k(x).$

Theorem 3.3: let $f \in C^{k+1}[a,b]$ and let $P(x)$ be the Lagrange Interpolating polynomial degree k in $[a,b]$. Then for each $x \in [a,b]$, there exists $\xi(x)$ in (a,b) such that

$f(x) = P(x) + \frac{f^{(k+1)}(\xi(x))}{(k+1)!}(x-x_0)(x-x_1)...(x-x_k).$

Neville's Method:

$P_{0,1,\ldots,k}(x) = \frac{(x-x_j)P_{0,1,\ldots,j-1,j+1,\ldots,k}(x)-(x-x_i)P_{0,1,\ldots,i-1,i+1,\ldots,k}(x)}{x_i-x_j}$
for any $i, j, i \neq j$.

Hermite Polynomial:

Given $(x_0, y_0), (x_1, y_1), ..., (x_k, y_k)$ and $(x_0, y_0'), (x_1, y_1'), ..., (x_k, y_k')$.

$\widetilde{H}(x) = \sum_{i=0}^{k} y_i H_i(x) + \sum_{i=0}^{k} y_i' \hat{H}_i(x)$.

where $H_i(x) = [1 - 2(x - x_i)L_i'(x_i)]L_i^2(x)$,

$\hat{H}_i(x) = (x - x_i)L_i^2(x)$,

and $L_i(x) = \frac{(x-x_0)(x-x_1)...(x-x_{i-1})(x-x_{i+1})...(x-x_k)}{(x_i-x_0)(x_i-x_1)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_k)}$.

Moreover, if $f \in C^{2k+2}[a + b]$, then

$f(x) = \widetilde{H}(x) + \frac{(x-x_0)^2...(x-x_k)^2}{(2k+2)!} f^{(2k+2)}(\xi)$ for some $\xi \in (a, b)$.

3. Chapter 4

Numerical Differentiation:

Two-point formula:
$f'(x_0) = \frac{f(x_0+h)-f(x_0)}{h} - \frac{h}{2} f''(\xi)$, where $\xi \in (x_0, x_0 + h)$.

Three-point formula:
$f'(x_0) = \frac{-3f(x_0)+4f(x_0+h)-f(x_0+2h)}{2h} + \frac{h^2}{3} f^{(3)}(\xi)$, where $\xi \in (x_0, x_0 + 2h)$.

Three-point formula:
$f'(x_0) = \frac{f(x_0+h)-f(x_0-h)}{2h} - \frac{h^2}{6} f^{(3)}(\xi)$, where $\xi \in (x_0 - h, x_0 + h)$.

Five-point formula:
$f'(x_0) = \frac{f(x_0-2h)-8f(x_0-h)+8f(x_0+h)-f(x_0+2h)}{12h} + \frac{h^4}{30} f^{(5)}(\xi)$, where $\xi \in (x_0 - 2h, x_0 + 2h)$.

Richardson Extrapolation:

If we know the approximation of M is in the form
$M = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + ...$

Then $N_i(h)$ (which is the diagonal entries in the table) can be obtain from the formula:

$N_i(h) = N_{i-1}(\frac{h}{2}) + \frac{N_{i-1}(\frac{h}{2})-N_{i-1}(h)}{2^{i-1}-1}$.

If we know the approximation of M is in the form
$M = N(h) + K_1h^2 + K_2h^4 + K_3h^6 + ...$

Then $N_i(h)$ (which is the diagonal entries in the table) can be obtain from the formula:

$$N_i(h) = N_{i-1}(\tfrac{h}{2}) + \frac{N_{i-1}(\frac{h}{2}) - N_{i-1}(h)}{4^{i-1}-1}.$$

Element of Numerical Integration:

Midpoint Rule:
$\int_a^b f(x)dx = (b-a)f(\frac{a+b}{2}) + \frac{h^3}{3}f''(\xi), \xi \in (a,b).$

Trapezoidal Rule:
$\int_a^b f(x)dx = \frac{h}{2}(f(a)+f(b)) - \frac{h^3}{12}f''(\xi), \quad \xi \in (a,b)$, where $\quad h = b-a,.$

Simpson's Rule:
$\int_a^b f(x)dx = \frac{h}{3}(f(a)+4f(\frac{a+b}{2})+f(b)) - \frac{h^5}{90}f^{(4)}(\xi), \quad \xi \in (a,b)$, where $\quad h = \frac{b-a}{2}.$

Composite Numerical Integration:

Error term of composite Trapezoidal Rule of $\int_a^b f(x)dx$:
Error term = $\frac{b-a}{12}h^2f''(\xi), \quad \xi \in (a,b)$, where h is step size.

Error term of composite Simpson's Rule of $\int_a^b f(x)dx$:
Error term = $\frac{b-a}{180}h^4f^{(4)}(\xi), \quad \xi \in (a,b)$, where h is step size.

4. Chapter 5

Approximation to the First Degree Ordinary Differential Equations:
$y' = f(t,y)$

Euler's Method
$w_i = w_{i-1} + h * f(t_{i-1}, w_{i-1}).$

Midpoint Method
$w_i = w_{i-1} + h * f(t_{i-1} + \frac{h}{2}, w_{i-1} + \frac{h}{2}f(t_{i-1}, w_{i-1})).$

Modified Euler Method
$w_i = w_{i-1} + \frac{h}{2} * (f(t_{i-1}, w_{i-1}) + f(t_{i-1} + h, w_{i-1} + hf(t_{i-1}, w_{i-1}))).$

Heun's Method
$w_i = w_{i-1} + \frac{h}{4} * (f(t_{i-1}, w_{i-1}) + 3f(t_{i-1} + \frac{2}{3}h, w_{i-1} + \frac{2}{3}hf(t_{i-1}, w_{i-1}))).$

Error bound for Euler's method of $y(t_i) - w_i$:
Error bound = $\frac{hM}{2L}[e^{L(t_i-a)} - 1],$

where L is a Lipschitz constant and $M = max\,|y''(t)|$ in the given interval $[a, b]$.