MATH 373: SOLUTION TO PRACTICE PROBLEM CLASS 19

1. Solution

1) For the uniqueness solution , we claim theorem 5.4 page 252.

Let $f(t,y)$ be $y'(t)$.

We need to show 1) $f(t,y)$ continuous.

2) $f(t,y)$ satisfies a Lipschitz condition in variable y on the given domain.

a) $f(t,y) = y * cos(t)$.

1) Since $g(t,y) = y$ is a continuous function and $h(t,y) = cos(t)$ is also a continuous function.

So $f(t,y)$, which is a product of continuous functions, is continuous.

2) We will apply theorem 5.3 (the condition to be Lipschitz) here.
We have $\left|\frac{\delta f}{\delta y}\right| = |cos(t)| \leq 1$.

So f satisfies Lipschitz condition with Lipschitz constant 1.

b) $f(t,y) = \frac{2}{t} * y + t^2 e^t$.

1) Since $g(t,y) = \frac{2}{t}y$ is a continuous function on $1 \leq t \leq 2$ and $h(t,y) = t^2 e^t$ is also a continuous function.

So $f(t,y)$, which is a product of continuous function, is continuous.

2) We will apply theorem 5.3. here.
We have $\left|\frac{\delta f}{\delta y}\right| = \left|\frac{2}{t}\right| \leq 2, \quad 1 \leq t \leq 2$.

So f satisfies Lipschitz condition with Lipschitz constant 2.

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2) a) The approximation using Euler Method:

Let $f(t, y) = y'(t) = 1 + (t - y)^2$.

Approximation term $= y + hy'$.

$$= y + hf(t, y).$$

$$= y + h(1 + (t - y)^2).$$

So we get the recurrence of the approximations:

$w_i = w_{i-1} + h(1 + (t_{i-1} - w_{i-1})^2)$.

In this problem $t_0 = 2,\ \ t_1 = 2.5,\ \ h = 0.5,\ \ w_0 = y_0 = 1$.

You plug everything in you get

$y(\frac{5}{2}) \approx w_1 = 2$.

$y(3) \approx w_2 = 2.625$.

b) Lipschitz condition:

$\left|\frac{\delta f}{\delta y}\right| = |(2(t - y)(-1))| \leq 2 * (3 - 1) = 4$.

Lipschitz constant = 4.

The above is a bit tricky since we don't know how big y could be. But we know y is increasing and $y(3)$ is not so big. so $max\,|t - y| = max(t) - min(y) = 3 - 1$.

c) $y(t) = t + \frac{1}{1-t}$.

Then $y'(t) = \frac{1}{(1-t)^2}$.

and $y''(t) = \frac{2}{(1-t)^3}$.

so $max_{t\in[2,3]}\,|y''(t)| = 2$ at $t = 2$.

d) Error bound $= \frac{hM}{2L}(e^{L(t_i - a)} - 1)$, (Theorem 5.9 in the book).

We have $h = 0.5,\ \ L = 4,\ \ M = 2$.

At $t = \frac{5}{2}$, error bound $= \frac{0.5*2}{2*4}(e^{4*0.5} - 1) \approx 0.798632$.

At $t = 3$, error bound $= \frac{0.5*2}{2*4}(e^{4*1} - 1) \approx 6.699768$.

(This formula did not give a really good error bound).

e) The exact solution to this problem is $y(t) = t + \frac{1}{1-t}$.

So $y(\frac{5}{2}) = \frac{5}{2} + \frac{1}{(1-\frac{5}{2})} \approx 1.83333$.

and $y(3) = 3 + \frac{1}{1-3} = 2.5$.

Actual Error at $t = \frac{5}{2}$ is $|2 - 1.83333| = 0.1666667$.

Actual Error at $t = 3$ is $|2.625 - 2.5| = 0.125$.

The error bound is pretty far off.