The On-Line Encyclopedia of Integer Sequences (OEIS)

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Introduction

OEIS is the database of integer sequences. For example, if I count some combinatorial object and get the first few terms to be

$$1, 3, 13, 63, \ldots.$$

I can find out more information from this website.

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We found out that the number of King walk from $(0, 0)$ to (n, n) grid is

$$a_n := \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n+k}{k}$$

The recurrence for a_n is

$$(n+2)a_n - (6n+9)a_{n-1} + (n+1)a_{n-2} = 0$$

Introduction

Neil Sloane, founder of OEIS

Time line

- Neil Sloane was born on Oct. 10, 1939
- 1964: start to collect integer sequences
- 1969: work at AT&T Bell labs (Now Nokia)
- 1973: publish ‘A handbook of integer sequences” which contains 2372 sequences.
- 1995: book version of “The encyclopedia of integer sequences” (5488 sequences)
- 1996: Internet version of “The encyclopedia of integer sequences”
- 2004: OEIS hits 100,000 sequences

Fun Fact

- There are about 315,271 sequences. The number of comments keeps increasing, and at present averages between 30 and 60 a day. Web traffic on all my web pages averages about 600,000 page-downloads per month.

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- Welcome page in Thai language!
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- 2D sequences or fraction sequences are in OEIS too. Just type them in!

Some famous/not so famous sequences

- A1: 0, 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14, 1, 5, . . .

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  (Number of groups of order *n*)
- A43: 2, 3, 5, 7, 13, 17, 19, . . .

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  (Mersenne primes)
- A326: 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, . . .

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  (Pentagonal numbers: $a(n) = \frac{n(3n-1)}{2}$)
- A787: 0, 1, 8, 11, 69, 88, 96, 101, 111, 181, 609, . . .

Neil Sloane's First Love

The sequence database was begun by Neil J. A. Sloane in early 1964 when he was a graduate student at Cornell University in Ithaca, NY. He had encountered a sequence of numbers while working on his dissertation, namely 1, 8, 78, 944, . . . (now entry A000435 in the OEIS), and was looking for a formula for the n-th term, in order to determine the rate of growth of the terms.

Neil Sloane's First Love

This sequence now has been expanded to more than quarter-million sequences and is expressible by the formula

$$(n-1)!\sum_{k=0}^{n-2}\frac{n^k}{k!}.$$

Sloane and John Riordan (1969) showed that this is the sum of the "total heights", taken over all labeled rooted tree with n vertices, divided by n.

Labeled Rooted Trees

Here we will follow Doron Zeilberger treatment, Going Back to Neil Sloane's FIRST LOVE (OEIS Sequence A435) , to this formula by Sloane.

Arthur Cayley famously proved that the number of labeled trees on n vertices is n^{n-2}, hence the number of labeled rooted tree is $n \cdot n^{n-2} = n^{n-1}$.

We will first prove this result (the method due to Andre Joyal) and expand the method to the result of Sloane and more.

Proof of Cayley's Formula

Theorem (Borchardt, 1860)

Let $r(n)$ be the number of labeled rooted trees with n vertices. We have

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Outline of the Proof

1. Let $R(x) := \sum_{n=0}^{\infty} \frac{r(n)}{n!} x^n$. Claim the functional equation:

$$R(x) = x \sum_{k=0}^{\infty} \frac{R(x)^k}{k!} = xe^{R(x)}.$$

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2. Apply Lagrange Inversion Theorem:
   If $R(x)$ and $\Phi(z)$ are formal power series which starting at x and z^0 respectively, then $R(x) = x\Phi(R(x))$ implies

$$[x^n]R(x) = \frac{1}{n}[z^{n-1}]\Phi(z)^n.$$

Proof of Sloane's Formula

Theorem

Let $s(n)$ be the sum of the total heights, taken over all labeled rooted tree with n vertices. We have

$$s(n) = n! \sum_{k=0}^{n-2} \frac{n^k}{k!}.$$

Proof of Sloane's Formula

Outline of the Proof

1. Consider $J_n(y) = \sum_T y^{TotalHeight(T)}$.
   Note $J_n(1) = r(n)$, number of rooted labeled tree.
   Then defined the formal power series

$$J(x, y) = \sum_{n=1}^{\infty} J_n(y) \frac{x^n}{n!}.$$

   Claimed the functional equation

$$J(x, y) = xe^{J(xy, y)}.$$

Proof of Sloane's Formula

2. It is too much too ask for formula for $J(x, y)$ explicitly. But for Sloane's formula, we only need $J_y(x, 1)$.
   This can be done by the chain rule, which gives us:

$$J_y(x, 1) = \frac{R(x)^2}{[1 - R(x)]^2}.$$

3. Apply General Lagrange Inversion Theorem:
   If $R(x)$ and $\Phi(z)$ are formal power series which starting at x and z^0 respectively, and $G(z)$ is yet another formal power series, then $R(x) = x\Phi(R(x))$ implies

$$[x^n]G(R(x)) = \frac{1}{n}[z^{n-1}]G'(z)\Phi(z)^n.$$