



The arithmetic-periodicity of cut for $\mathcal{C} = \{1, 2c\}$

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ABSTRACT

cut is a class of partition games played on a finite number of finite piles of tokens. Each version of cut is specified by a cut-set $\mathcal{C} \subseteq \mathbb{N}$. A legal move consists of selecting one of the piles and partitioning it into $d + 1$ nonempty piles, where $d \in \mathcal{C}$. No tokens are removed from the game. It turns out that the nim-set for any $\mathcal{C} = \{1, 2c\}$ with $c \geq 2$ is arithmetic-periodic, which answers an open question of Dailly et al. (2020). The key step is to show that there is a correspondence between the nim-sets of cut for $\mathcal{C} = \{1, 6\}$ and the nim-sets of cut for $\mathcal{C} = \{1, 2c\}$, $c \geq 4$. The result easily extends to the case of $\mathcal{C} = \{1, 2c_1, 2c_2, 2c_3, \dots\}$, where $c_1, c_2, \dots \geq 2$.

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1. Partition games and the game cut

In this paper, we calculate the nim-sequence of a particular class of the game cut. This game is part of a larger class of *partition games*. Partition games are characterized by the fact that a legal move consists of selecting one of various piles and partitioning it into some smaller piles. In these games, no tokens are ever removed. Two notable examples are COUPLES ARE FOREVER and GRUNDY'S GAME. In the former, a player may partition any pile of size 3 or greater into two smaller piles. In the latter, a player may partition any pile into two smaller piles of *unequal* size. Incidentally, the nim-sequence for these games remains unknown.

For cut, the story is a bit different. Here each particular game of cut is defined by a set of positive integers, \mathcal{C} , called the *cut-set*. A legal move for the game with cut-set \mathcal{C} consists of selecting a pile and partitioning it into $d + 1$ nonempty parts, for some $d \in \mathcal{C}$. This is the same as “cutting” the pile d times.

Given the breadth of combinatorial game theory writ large [2], the authors are surprised to learn that cut has only been studied very recently in [3]. We will calculate the nim-sequence for cut when $\mathcal{C} = \{1, 2c\}$ for all $c \geq 2$. So for the games in this paper, a player may take any of the piles and split it into either two piles or $2c + 1$ piles. In general, a combinatorial game ends only when there are no legal moves remaining, so for these games, this is when all the tokens are in piles of size 1.

All of the games mentioned have two variants: *normal play* and *misère play*. In normal play, the last player to make a legal move wins, and in misère play, the last player loses. We only investigate the normal play variant.

The classic Sprague–Grundy Theorem says that, under normal play, each position in any of these games (or any impartial game) is equivalent to some one-pile position of nim. Hence we say that if a certain position is equivalent to a nim pile of size n , then it has *nim-value* of n . Fixing a cut-set \mathcal{C} , we write $\mathcal{G}(n)$ to denote the nim-value of the position of cut with a single pile of size n . Then $(\mathcal{G}(n))_{n \geq 1}$ is called the *nim-sequence* of the game with cut-set \mathcal{C} .

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Table 1
Some known results about cut.

Cut-set \mathcal{C}	Nim sequence	Proposition in [3]
$1 \notin \mathcal{C}$	$(0)^c(+1)$ where c is the smallest element in \mathcal{C}	6
$1 \in \mathcal{C}$ and \mathcal{C} contains only odd numbers	$(0, 1)$	3
$\{1, 2, 3\} \subseteq \mathcal{C}$	$(0)(+1)$ (i.e. $\mathcal{G}(n) = n - 1$)	7
$\mathcal{C} = \{1, 3, 2c\}$	$(0, 1)^c(+2)$	8

Of course, after just one move, any game of cut will have more than one pile. cut, like NIM, is *disjunctive*, which means each move only affects one of the existing piles. The Sprague–Grundy Theorem further tells us the nim-values only make sense if we calculate the nim-value of a position with multiple piles using the *nim-sum*. The nim-sum of x and y , denoted $x \oplus y$ is the binary sum modulo 2. For example, $9 \oplus 11 = 1001_2 \oplus 1011_2 = 10_2 = 2$. Then to calculate the nim-sequence for cut, or any combinatorial game, we proceed recursively:

Definition 1. Given a pile of size n , an *option* O_n is a sequence of piles obtainable via a single legal move on that pile. In other words, O_n is a particular partition of n with $d + 1$ parts for some $d \in \mathcal{C}$. Formally, $O_n = (h_0, h_1, \dots, h_d)$ where $1 \leq h_i$ and $h_0 + h_1 + \dots + h_d = n$. Let \mathcal{O}_n denote the set of all options O_n .

Definition 2. The nim-value of the option $O_n = (h_0, h_1, \dots, h_d)$ is

$$\mathcal{G}(O_n) = \mathcal{G}(h_0) \oplus \mathcal{G}(h_1) \oplus \dots \oplus \mathcal{G}(h_d).$$

Finally the Sprague–Grundy Theorem tells us that the nim-value $\mathcal{G}(n)$ is given as

$$\mathcal{G}(n) = \text{mex}\{\mathcal{G}(O_n) \mid O_n \in \mathcal{O}_n\},$$

where $\text{mex}(S)$ is the least natural number missing from S . For example, $\text{mex}(0, 3, 3, 1, 5) = 2$. For readers unfamiliar with the nim-sum and the mex function, we refer them to [1]. Finally, we note that the nim-sequence of a game encodes its winning strategy. The positions with nim-value 0 are called \mathcal{P} -positions, meaning that the previous player can win, and the other positions are called \mathcal{N} -positions, meaning that the next player can win. In the latter case, the next player wins by making a move which turns the current \mathcal{N} -position into a \mathcal{P} -position.

2. Known results and our contribution

Let us summarize the results in [3] to give the reader some idea about the current state of affairs. We found that paper to be rather clear, and we recommend working through these examples as an initial exercise. They also elaborate on some properties of nim-sum that we only mention briefly in this paper. Table 1 shows the known nim-sequences of various versions of cut.

The notation $(0, 1)^3(+2)$, for example, denotes the sequence

$$0, 1, 0, 1, 0, 1, 2, 3, 2, 3, 2, 3, 4, 5, 4, 5, 4, 5, \dots$$

In this case, we would say that the sequence is arithmetic-periodic with period 6 and *saltus* 2. Formally, a sequence $\mathcal{G}(n)$ is *arithmetic-periodic* if there is a period P and a saltus $s > 0$ such that for all $n > 0$,

$$\mathcal{G}(n + P) = \mathcal{G}(n) + s.$$

One of the cases that [3] conjectured to also have an arithmetic-periodic nim-sequence is $\mathcal{C} = \{1, 2c\}$ for any $c \geq 2$. Most of the rest of this paper will be the proof of that. The proof is quite long, and we owe some gratitude to our computer friend. It helped us to make many important observations. In particular, we will prove:

Theorem 3 (Main Target). The nim-sequence of the game cut with cut-set $\mathcal{C} = \{1, 2c\}$ for any $c \geq 2$ is precisely

$$(0, 1)^c(2, 3)^c, 1, 4, (5, 4)^{c-1}, (3, 2)^c(4, 5)^c(6, 7)^c(+8).$$

This is Open Problem 1 in [3], though they did prove it for the cases $c = 2, 3, 4, 5$. (Refer to Table 1, Corollary 15, and the discussion preceding Open Problem 1.)

Our nim-sequence is arithmetic-periodic with period $12c$ and saltus 8. Arithmetic-periodic nim-sequences with a power of 2 as saltus admit a division algorithm style decomposition of the nim-values of options. The following is essentially Lemmas 10 and 11 in [3].

Table 2
The first $12c$ nim-values of cut for $c = \{1, 2c\}$, $c \geq 3$.

0	1	0	1	0	1	...	0	1
2	3	2	3	2	3	...	2	3
1	4	5	4	5	4	...	5	4
3	2	3	2	3	2	...	3	2
4	5	4	5	4	5	...	4	5
6	7	6	7	6	7	...	6	7

Proposition 4. Suppose there exists a period P , a saltus $s = 2^t$, $t > 1$ such that for all $n \leq P$,

$$\mathcal{G}(n) < s,$$

and for all $n > P$

$$\mathcal{G}(n) = \mathcal{G}(n - P) + s.$$

Let $h_i = k_iP + r_i \leq N$ and $h'_i = k_iP + r'_i \leq N$ where $1 \leq r_i, r'_i \leq P$, for all $0 \leq i \leq d$. Suppose $O_n = (h_0, h_1, \dots, h_d)$ and $O'_n = (h'_0, h'_1, \dots, h'_d)$. Then

$$\mathcal{G}(O_n) = \mathcal{G}(O'_n) \text{ if and only if } \mathcal{G}((r_0, r_1, \dots, r_d)) = \mathcal{G}((r'_0, r'_1, \dots, r'_d)).$$

Proof. By the arithmetic-periodic property,

$$\mathcal{G}(h_i) = \mathcal{G}(k_iP + r_i) = k_i s + \mathcal{G}(r_i).$$

Then

$$\begin{aligned} \mathcal{G}(O_n) &= \mathcal{G}(h_0) \oplus \mathcal{G}(h_1) \oplus \dots \oplus \mathcal{G}(h_d) \\ &= (k_0 \oplus k_1 \oplus \dots \oplus k_d)s + [\mathcal{G}(r_0) \oplus \mathcal{G}(r_1) \oplus \dots \oplus \mathcal{G}(r_d)] && \text{since } \mathcal{G}(r_i) < s = 2^t \\ &= (k_0 \oplus k_1 \oplus \dots \oplus k_d)s + \mathcal{G}((r_0, r_1, \dots, r_d)) \end{aligned}$$

Similarly, $\mathcal{G}(O'_n) = (k_0 \oplus k_1 \oplus \dots \oplus k_d)s + \mathcal{G}((r'_0, r'_1, \dots, r'_d)) \quad \square$

In the proofs that follow, we will want to find different options with the same nim-values. Proposition 4 allows us to restrict many calculations to the first period of $\mathcal{G}(n)$.

This paper is accompanied by the Maple program *PartitionGame* which can be found on the second author’s website: www.thotsaporn.com. The purpose of the program, apart from generating the data shown in this paper, is to search for the hidden structure of the problem. This turned out to be the patterns in the Nim-sets described in the subsequent sections.

3. Patterns in the sequence and Nim-sets

Ideally the proof would be nice and compact, but we cannot avoid using some properties from the nim-sequence itself. Let us write just the first period in a table of 6 ‘rows’ (see Table 2).

Each row has $2c$ entries with alternating entries, except that the first entry of the third row is 1 (marked in bold), for a total of $2c \cdot 6 = 12c$ entries for each period. The following observations will be very useful at several key stages of the proof. They follow directly from Theorem 3. Thus they are assumed to be true for the base cases $c = 2, 3$, and will be used recursively in many of the proofs of the next section.

Excepting the first entry of the third row, note that the numbers in each row do not just alternate, they do so by nim-adding 1. More precisely,

$$\mathcal{G}(n + 1) = \mathcal{G}(n) \oplus 1 \text{ for } n \not\equiv 0 \pmod{2c} \text{ and } n \not\equiv 4c + 1 \pmod{12c} \tag{ob1}$$

The remaining observations show us how to find different partitions of the same number (i.e., two moves from the same position) which have the same nim-value. This one is about the final entries of the rows of the first period.

$$\mathcal{G}(2c) \oplus \mathcal{G}(12c) = \mathcal{G}(4c) \oplus \mathcal{G}(10c) = \mathcal{G}(6c) \oplus \mathcal{G}(8c) = 6 \tag{ob2}$$

The rest are not restricted to the first period, and they all leverage the fundamental fact about the nim-sum that $k \oplus k = 0$ for all k .

$$\mathcal{G}(n) \oplus \mathcal{G}(n) \oplus \mathcal{G}(m + 1) \oplus \mathcal{G}(m + 1) = 0 = \mathcal{G}(n + 1) \oplus \mathcal{G}(n + 1) \oplus \mathcal{G}(m) \oplus \mathcal{G}(m) \tag{ob3.1}$$

Here note that the nim-value of the last entry in each row of each period is the same as two entries prior:

$$\mathcal{G}(n) \oplus \mathcal{G}(n) \oplus \mathcal{G}(2ac) = \mathcal{G}(n + 1) \oplus \mathcal{G}(n + 1) \oplus \mathcal{G}(2ac - 2) \tag{ob3.2}$$

Table 3
The first 17 nim-sets of cut for $c = \{1, 6\}$.

n	1	2	3	4	5	6	7	8	9	10
$\mathcal{N}(n, 2, \{1, 6\})$	–	{0}	{1}	{0}	{1}	{0}	{1}	{0,2}	{1,3}	{0,2}
$\mathcal{N}(n, 7, \{1, 6\})$	–	–	–	–	–	–	{0}	{1}	{0}	{1}
$\mathcal{G}_{\{1,6\}}(n)$	0	1	0	1	0	1	2	3	2	3

n	11	12	13	14	15	16	17
$\mathcal{N}(n, 2, \{1, 6\})$	{1,3}	{0,2}	{3}	{0,1,2}	{0,1,3,4}	{0,1,2,5}	{0,1,3,4}
$\mathcal{N}(n, 7, \{1, 6\})$	{0}	{1}	{0,2}	{1,3}	{0,2}	{1,3}	{0,2}
$\mathcal{G}_{\{1,6\}}(n)$	2	3	1	4	5	4	5

Table 4
The first 30 nim-sets of cut for $c = \{1, 6\}$, $p = 2$, decomposed into alternating subsequences.

n	1	2	3	4	5	6	7	8	9	10
$\mathcal{N}(n, 2, \{1, 6\})$	–	0	1	0	1	0	1	0	1	0

n	11	12	13	14	15	16	17	18	19
$\mathcal{N}(n, 2, \{1, 6\})$	1	0		2	3	2	3	2	
	3	2	3	1	0	1	0	1	0
				0	1	0	1	0	1
					4	5	4	5	4

n	20	21	22	23	24	25	26	27	28	29	30
$\mathcal{N}(n, 2, \{1, 6\})$											
	0	1	0	1	0						
	5	4	5	4	5						
	3	2	3	2	3	2	3	2	3	2	3
		6	7	6	7	6	7	6	7	6	7
							0				
							1	0	1	0	1
							4	5	4	5	4
									0	1	0

Similarly, the first entry in any but the third row of any period is the same as two entries later:

$$\mathcal{G}(n + 1) \oplus \mathcal{G}(n + 1) \oplus \mathcal{G}(2ac + 1) = \mathcal{G}(n) \oplus \mathcal{G}(n) \oplus \mathcal{G}(2ac + 3), \quad a \not\equiv 2 \pmod 6 \tag{ob3.3}$$

Finally, the second entry in the third row of each period is the same as two entries later:

$$\mathcal{G}(n + 1) \oplus \mathcal{G}(n + 1) \oplus \mathcal{G}(2ac + 2) = \mathcal{G}(n) \oplus \mathcal{G}(n) \oplus \mathcal{G}(2ac + 4), \quad a \equiv 2 \pmod 6 \tag{ob3.4}$$

Since our version of cut has two distinct types of options, it makes sense to consider the effect of each one on the nim-sequence separately.

Definition 5. The nim-set, $\mathcal{N}(n, p, C)$, is the set of nim values that arise from breaking n tokens into p piles. Formally,

$$\mathcal{N}(n, p, C) = \{\mathcal{G}(O_n) \mid O_n = (h_1, h_2, h_3, \dots, h_p) \text{ where } h_1 + h_2 + \dots + h_p = n\}.$$

Note that each nim-value in this set, $\mathcal{G}(O_n) = \mathcal{G}(h_1) \oplus \mathcal{G}(h_2) \oplus \dots \oplus \mathcal{G}(h_p)$, is still calculated recursively according to the actual rules of the game, that is, using the whole cut-set C . As an initial observation, note that $\mathcal{N}(n, 1, C) = \{\mathcal{G}_C(n)\}$. Furthermore, if $C = \{c_1, c_2, \dots, c_p\}$, then $\mathcal{G}_C(n) = \text{mex}\{\cup_{i=1}^p \mathcal{N}(n, c_i + 1, C)\}$.

We will use $c = 3$ as a motivating example in the next section, and as the base case for our eventual induction. Thus let $C = \{1, 6\}$ and so $p = 2$ or 7 . Table 3 shows the initial terms in the nim-sequence, decomposed into their nim-sets.

4. Entering and exiting partitions

When we further analyze the sequence of nim-sets for $p = 2$, we find that it is made up of alternating subsequences. See Table 4.

This is quite striking! There are underlying subsequences which alternate between a and $a \oplus 1$. Furthermore, these subsequences enter and exit the sequence at very particular partitions. Here are the partitions where these subsequences begin and end:

- The first subsequence begins at $O_2 = (1, 1)$ and ends at $O_{12} = (6, 6)$

- The second subsequence begins at $O_8 = (7, 1)$ and ends at $O_{18} = (12, 6)$
- The third subsequence begins at $O_{14} = (13, 1)$ and ends at $O_{19} = (13, 6)$
- The fourth subsequence begins at $O_{14} = (7, 7)$ and ends at $O_{24} = (12, 12)$
- The fifth subsequence begins at $O_{15} = (14, 1)$ and ends at $O_{24} = (18, 6)$
- The sixth subsequence begins at $O_{20} = (13, 7)$ or $O_{20} = (19, 1)$ and ends at $O_{30} = (24, 6)$
- The seventh subsequence begins at $O_{21} = (14, 7)$ and ends at $O_{30} = (18, 12)$
- The eighth subsequence begins **and** ends at $O_{26} = (13, 13)$
- The ninth subsequence begins at $O_{26} = (19, 7) \dots$
- The tenth subsequence begins at $O_{26} = (25, 1) \dots$
- The eleventh subsequence begins at $O_{28} = (14, 14) \dots$

Continuing in this fashion, we find that each of these subsequences begins at a partition whose parts are all either $1 \pmod{2c}$ or $(4c + 2) \pmod{12c}$, and ends at a partition whose parts are all either $0 \pmod{2c}$ or $(4c + 1) \pmod{12c}$. This motivates the following definition.

Definition 6. Fix a cut-set $C = \{1, 2c\}$. Call a number $n > 0$ an *innumber* if $n \equiv 1 \pmod{2c}$ or $n \equiv (4c + 2) \pmod{12c}$. Then call an option $O_n = \{h_0, \dots, h_d\}$ an *entering partition* if each h_i is an innumber. Similarly, call a number $n > 0$ an *outnumber* if $n \equiv 0 \pmod{2c}$ or $n \equiv (4c + 1) \pmod{12c}$. Then call an option $O_n = \{h_0, \dots, h_d\}$ an *exiting partition* if each h_i is an outnumber. Finally, call an option $O_n = \{h_0, \dots, h_d\}$ an *intermediate partition* if it is neither an entering nor an exiting partition.

Notice that the innumbers and outnumberers are the beginnings and ends of the rows from the table in Section 3. In fact, if we consider $(4c + 1) \pmod{12c}$ and then $(4c + 2) \pmod{12c}$ through $6c \pmod{12c}$ as two separate rows, this is precise.

Different partitions of the same number can give the same nim-value. This happens frequently. For example, the first subsequence arises from either of the following sequences of partitions:

$$(1, 1), (2, 1), (2, 2), (3, 2), (3, 3), (4, 3), (4, 4), (5, 4), (5, 5), (6, 5), (6, 6)$$

$$(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)$$

More generally, if a partition is not entering, then its nim-value can be obtained recursively. In fact, this is the case for any $p \geq 2$:

Lemma 7. Assume [Theorem 3](#) holds up to n . Suppose the option $O_{n+1} = (h_1, \dots, h_p)$ is not an entering partition. Then there is an option O'_n so that $\mathcal{G}(O_{n+1}) = \mathcal{G}(O'_n) \oplus 1$. Similarly, if $O_n = (j_1, \dots, j_p)$ is not an exiting partition, then there is an option O'_{n+1} so that $\mathcal{G}(O'_{n+1}) = \mathcal{G}(O_n) \oplus 1$.

Proof. Choose i where h_i is not an innumber. Then $h_i - 1$ is not an outnumber. Let $O'_n = (h_1, \dots, h_i - 1, \dots, h_p)$.

$$\begin{aligned} \mathcal{G}(O_{n+1}) &= \mathcal{G}(h_1) \oplus \dots \oplus \mathcal{G}(h_i) \oplus \dots \oplus \mathcal{G}(h_p) \\ &= \mathcal{G}(h_1) \oplus \dots \oplus [\mathcal{G}(h_i - 1) \oplus 1] \oplus \dots \oplus \mathcal{G}(h_p) && \text{(ob1)} \\ &= \mathcal{G}(O'_n) \oplus 1 \end{aligned}$$

In the second case, choose i so that j_i is not an outnumber. Then define $O'_{n+1} = (j_1, \dots, j_i + 1, \dots, j_p)$, and apply the same calculation. \square

In this proof, the use of (ob1) is valid because of our induction hypothesis that [Theorem 3](#) holds up to n . The other observations from the previous section, which also follow from [Theorem 3](#), are used similarly in the rest of the proofs of this section.

Extending this calculation, the nim-value of every non-entering partition is either the same as the greatest entering partition less than or equal to it, or off by $\oplus 1$. To be precise, we have the following definition and corollary.

Definition 8. Suppose $O_n = (h_1, \dots, h_p)$ is a partition of n . For each $1 \leq i \leq p$, let $\lfloor h_i \rfloor$ be the greatest innumber less than or equal to h_i . Then call $\lfloor O_n \rfloor = (\lfloor h_1 \rfloor, \dots, \lfloor h_p \rfloor)$ the *floor* of O_n . Similarly, let $\lceil h_i \rceil$ be the least outnumber greater than or equal to h_i , and call $\lceil O_n \rceil = (\lceil h_1 \rceil, \dots, \lceil h_p \rceil)$ the *ceiling* of O_n . Write $O_n \sim O'_{n'}$ if $\lfloor O_n \rfloor = \lfloor O'_{n'} \rfloor$. For technical reasons, in the case that $\lfloor h_1 \rfloor \equiv \lfloor h_2 \rfloor \equiv \lfloor h_3 \rfloor \equiv 2 \pmod{2c}$, we set $\lfloor (h_1, h_2, h_3) \rfloor = (\lfloor h_1 \rfloor - 1, \lfloor h_2 \rfloor - 1, \lfloor h_3 \rfloor)$.

Note that \sim defines an equivalence relation on all partitions, and that the floor and ceiling partitions are always entering and exiting partitions, respectively. Then we have the following. To check the exceptional case, note that

$$\mathcal{G}(12cq_1 + 4c + 2, 12cq_2 + 4c + 2, 12cq_3 + 4c + 2) = \mathcal{G}(12cq_1 + 4c + 1, 12cq_2 + 4c + 1, 12cq_3 + 4c + 2).$$

Corollary 9. Assume [Theorem 3](#) holds up to $\max\{n, n'\}$. Suppose $O_n \sim O'_n$. If $n \equiv n' \pmod 2$, then $\mathcal{G}(O_n) = \mathcal{G}(O'_n)$. Otherwise, $\mathcal{G}(O_n) = \mathcal{G}(O'_n) \oplus 1$.

Conversely, suppose O_n is a partition of n , $\lfloor O_n \rfloor$ is a partition of a , and $\lceil O_n \rceil$ is a partition of b . If $a \leq n' \leq b$, and $n' \equiv n \pmod 2$, then there is a partition O'_n of n' with the same number of parts as O_n satisfying $\mathcal{G}(O'_n) = \mathcal{G}(O_n)$.

Using only [Lemma 7](#), it might seem that the sixth subsequence 3, 4, 3, 4, . . . which begins at $O_{20} = (13, 7)$ would end at $O_{25} = (13, 12)$. But we saw above that it ends at O_{30} . This is because the exiting partition $O_{25} = (13, 12)$ has the same nim-value as, for example, the non-exiting partition $O_{25} = (19, 6)$, and thus [Lemma 7](#) implies that the subsequence continues on to $O_{30} = (24, 6)$. The next result shows us that, for $p \geq 4$, this replacement is always possible.

Lemma 10. Assume [Theorem 3](#) holds up to n . Suppose $p \geq 4$ and $C = \{1, 2c\}$, $c \geq 2$. Then each exiting partition in $\mathcal{N}(n, p, C)$ has the same nim-value as some non-exiting partition in $\mathcal{N}(n, p, C)$.

So for $p \geq 4$, once one of these subsequences starts, it never ends. Formally:

Corollary 11. Assume [Theorem 3](#) holds up to n . For $p \geq 4$, $c \geq 2$, if $v \in \mathcal{N}(n, p, \{1, 2c\})$, then $v \oplus 1 \in \mathcal{N}(n + 1, p, \{1, 2c\})$.

It may seem overly general that we write $p \geq 4$ rather than $p = 2c + 1$, since $C = \{1, 2c\}$. However, the proof of the main result will use an induction that involves all values of $p \geq 1$.

Proof of Lemma 10. Let $O_n = (h_1, h_2, h_3, h_4, \dots, h_p)$ be an exiting partition. We will find a non-exiting partition $O'_n = (h'_1, h'_2, h'_3, h'_4, \dots, h'_p)$ with the same nim-value. In fact, we will only alter the remainders of h_1, h_2, h_3, h_4 modulo $12c$, so [Proposition 4](#) assures us that we only need to check that the nim-sum of the nim-values of these remainders remains unaltered. The seven outnumbers less than or equal to $12c$ are precisely $2c, 4c, 4c + 1, 6c, 8c, 10c, 12c$.

Case 1: h_1, h_2, h_3, h_4 are all congruent to $0 \pmod{2c}$.

If there is a repeated value, then apply [\(ob3.2\)](#) to get a non-exiting partition with the same nim-value. If not, then by the pigeonhole principle, one of the pairs $\{2c, 12c\}$, $\{4c, 10c\}$, $\{6c, 8c\}$ must be present, so we can apply [\(ob2\)](#) to get a repeated value.

Case 2: $h_1 = 4c + 1$ and $h_2, h_3, h_4 \equiv 0 \pmod{2c}$.

Again, if there is a repeated value, we can apply [\(ob3.2\)](#), so assume not.

If one of the values $4c, 6c$ or $12c$ are present, we can adjust as follows to obtain a non-exiting partition:

$$\mathcal{G}(4c + 1) \oplus \mathcal{G}(4c) = 2 = \mathcal{G}(6c + 1) \oplus \mathcal{G}(2c)$$

$$\mathcal{G}(4c + 1) \oplus \mathcal{G}(6c) = 5 = \mathcal{G}(8c + 1) \oplus \mathcal{G}(2c)$$

$$\mathcal{G}(4c + 1) \oplus \mathcal{G}(12c) = 6 = \mathcal{G}(10c) \oplus \mathcal{G}(6c + 1)$$

Otherwise (h_2, h_3, h_4) must be $(2c, 8c, 10c)$. In this case, apply

$$\mathcal{G}(4c + 1) \oplus \mathcal{G}(2c) \oplus \mathcal{G}(8c) \oplus \mathcal{G}(10c) = 7 = \mathcal{G}(6c + 1) \oplus \mathcal{G}(6c) \oplus \mathcal{G}(6c) \oplus \mathcal{G}(6c).$$

Case 3: Exactly two or three of h_1, h_2, h_3, h_4 are $h_i = 4c + 1$.

Apply [\(ob3.2\)](#).

Case 4: $h_1 = h_2 = h_3 = h_4 = 4c + 1$.

Apply [\(ob3.1\)](#). \square

Applying the same technique, we find that, for $p \geq 4$, these subsequences really only begin at entering partitions which have a part of size 1.

Lemma 12. Assume [Theorem 3](#) holds up to n . Suppose $p \geq 4$ and $C = \{1, 2c\}$, $c \geq 2$. Then each entering partition in $\mathcal{N}(n, p, C)$ without a part of size 1 has the same nim-value as some non-entering partition in $\mathcal{N}(n, p, C)$.

Proof. Suppose $O_n = (h_1, h_2, h_3, h_4, \dots, h_p)$ is an entering partition so that $h_1, h_2, h_3, h_4 > 1$. We will find a non-entering partition $O'_n = (h'_1, h'_2, h'_3, h'_4, \dots, h'_p)$ with the same nim-value. As in the previous proof, we will only alter the remainders of h_1, h_2, h_3, h_4 modulo $12c$, so [Proposition 4](#) ensures that we only need to check that the nim-sum of the nim-values of these remainders remains unaltered. The seven innumbers less than or equal to $12c$ are precisely $1, 2c + 1, 4c + 1, 4c + 2, 6c + 1, 8c + 1, 10c + 1$. In some of the cases below, we replace h_i with $h_i - 1$. So even though it may be that $h_i \equiv 1 \pmod{2c}$, we do require $h_i > 1$.

Case 1: $h_1 = h_2$ and $h_3 = h_4$.

Apply [\(ob3.1\)](#).

Case 2: $h_1 = h_2$ and $h_3 \neq h_4$.

Then one of h_3 or h_4 is not $4c + 1$, say h_3 . Apply [\(ob3.3\)](#) if $h_3 \equiv 1 \pmod{2c}$ or [\(ob3.4\)](#) if $h_3 = 4c + 2$.

Case 3: h_1, h_2, h_3, h_4 are all different.

Case 3.1: All h_i are 1 mod $2c$. We will show by subcases how to find an entering partition with the same nim-value that has repeated values. Then Case 1 or Case 2 applies.

Case3.1a: $\{h_1, h_2\} = \{8c + 1, 10c + 1\}$ then $h_3 \in \{1, 2c + 1, 4c + 1, 6c + 1\}$. We will be able to apply one of the following to obtain a repeated value of either $8c + 1$ or $10c + 1$:

$$\begin{aligned} \mathcal{G}(10c + 1) \oplus \mathcal{G}(1) &= 6 = \mathcal{G}(8c + 1) \oplus \mathcal{G}(2c + 1) \\ \mathcal{G}(10c + 1) \oplus \mathcal{G}(4c + 1) &= 7 = \mathcal{G}(8c + 1) \oplus \mathcal{G}(6c + 1) \end{aligned}$$

Case3.1b: $h_1, h_2, h_3 \in \{1, 2c + 1, 4c + 1, 6c + 1\}$. Then apply:

$$\mathcal{G}(6c + 1) \oplus \mathcal{G}(1) = 3 = \mathcal{G}(2c + 1) \oplus \mathcal{G}(4c + 1)$$

Case 3.2: $h_1 = 4c + 2$.

Case 3.2a: $h_1 = 4c + 2, h_2 = 4c + 1$. Apply

$$\mathcal{G}(4c + 2) \oplus \mathcal{G}(4c + 1) = 5 = \mathcal{G}(8c + 2) \oplus \mathcal{G}(1)$$

Case 3.2b: $h_1 = 4c + 2, h_2 = 6c + 1$. Apply

$$\mathcal{G}(4c + 2) \oplus \mathcal{G}(6c + 1) = 7 = \mathcal{G}(8c + 2) \oplus \mathcal{G}(2c + 1)$$

Case 3.2c: $(h_1, h_2, h_3, h_4) = (4c + 2, 1, 8c + 1, 10c + 1)$ or $(h_1, h_2, h_3, h_4) = (4c + 2, 2c + 1, 8c + 1, 10c + 1)$. Apply one of the calculations in Case 3.1a to obtain a repeated value of $8c + 1$.

Case 3.2d: $(h_1, h_2, h_3, h_4) = (4c + 2, 1, 2c + 1, 10c + 1)$. Apply

$$\mathcal{G}(4c + 2) \oplus \mathcal{G}(1) \oplus \mathcal{G}(2c + 1) \oplus \mathcal{G}(10c + 1) = 0 = \mathcal{G}(10c) \oplus \mathcal{G}(6c) \oplus \mathcal{G}(4) \oplus \mathcal{G}(1)$$

Case 3.2e: $(h_1, h_2, h_3, h_4) = (4c + 2, 1, 2c + 1, 8c + 1)$. Apply

$$\mathcal{G}(4c + 2) \oplus \mathcal{G}(1) \oplus \mathcal{G}(2c + 1) \oplus \mathcal{G}(8c + 1) = 2 = \mathcal{G}(8c) \oplus \mathcal{G}(3c + 1) \oplus \mathcal{G}(3c + 1) \oplus \mathcal{G}(3) \quad \square$$

Putting together everything we now know about entering and exiting partitions, we obtain a remarkable recursion on the nim-sets.

Theorem 13. Assume Theorem 3 holds up to n . For $p \geq 4$ and $c \geq 2$,

$$\mathcal{N}(n + 1, p + 1, \{1, 2c\}) = \mathcal{N}(n, p, \{1, 2c\}).$$

Proof. Fix $p \geq 4$. We proceed by induction on n . The base case is $n = p$, and we have

$$\begin{aligned} \mathcal{N}(p, p, \{1, 2c\}) &= \{\mathcal{G}((1_1, 1_2, \dots, 1_p))\} \\ &= \{0\} \\ &= \{\mathcal{G}((1_1, 1_2, \dots, 1_{p+1}))\} = \mathcal{N}(p + 1, p + 1, \{1, 2c\}) \end{aligned}$$

Next, fix $n > p$, and suppose $\mathcal{N}(n - 1, p, \{1, 2c\}) = \mathcal{N}(n, p + 1, \{1, 2c\})$. Consider any O_n with p parts. Appending a pile of size 1 does not change the nim-value of an option, so we can define O'_{n+1} with $p + 1$ parts accordingly. Hence $\mathcal{N}(n, p, \{1, 2c\}) \subset \mathcal{N}(n + 1, p + 1, \{1, 2c\})$.

Next suppose $O_{n+1,p+1}$ is a partition of $n + 1$ with $p + 1$ parts. If one of the parts is precisely 1, then removing this part gives us a partition of n with p parts that has the same nim-value.

Assume $O_{n+1,p+1}$ has no parts of size 1. If it is an entering partition, apply Lemma 12 to obtain a non-entering partition $O'_{n+1,p+1}$ of $n + 1$ with $p + 1$ parts so that $\mathcal{G}(O'_{n+1,p+1}) = \mathcal{G}(O_{n+1,p+1})$. (If $O_{n+1,p+1}$ is already non-entering, then set $O'_{n+1,p+1} = O_{n+1,p+1}$.) Next apply Lemma 7 to obtain a partition $O_{n,p+1}$ of n with $p + 1$ parts so that $\mathcal{G}(O'_{n+1,p+1}) = \mathcal{G}(O_{n,p+1}) \oplus 1$. Then by the induction hypothesis, there is some partition $O_{n-1,p}$ of $n - 1$ with p parts satisfying $\mathcal{G}(O_{n-1,p}) = \mathcal{G}(O_{n,p+1})$. Next if $O_{n-1,p}$ is an exiting partition, apply Lemma 10 to replace it with a non-exiting partition $O'_{n-1,p}$ with the same nim-value (and p parts). Finally apply Lemma 7 to obtain a partition $O_{n,p}$ of n with p parts satisfying $\mathcal{G}(O_{n,p}) = \mathcal{G}(O'_{n-1,p}) \oplus 1$. Putting all this together, we obtain the following equation

$$\mathcal{G}(O_{n+1,p+1}) = \mathcal{G}(O'_{n+1,p+1}) = \mathcal{G}(O_{n,p+1}) \oplus 1 = \mathcal{G}(O_{n-1,p}) \oplus 1 = \mathcal{G}(O'_{n-1,p}) \oplus 1 = \mathcal{G}(O_{n,p}),$$

And hence $\mathcal{N}(n + 1, p + 1, \{1, 2c\}) \subset \mathcal{N}(n, p, \{1, 2c\})$. \square

5. Proof of Theorem 3

As mentioned above, [3] verified Theorem 3 for $c = 2, 3, 4, 5$. Our strategy is to show that for $c > 3$, the nim-sequence for $\mathcal{C} = \{1, 2c\}$ is a sort of ‘shifted-expanded version’ of the nim-sequence for $\mathcal{C} = \{1, 6\}$. In fact, we will derive this from the same correspondence on the associated nim-sets. Looking at the statement of the theorem, we might think that there

are a few ways we might define the correspondence. However, when we look at a few examples of individual nim-sets, we see that our choices are limited. Let us compare the Maple generated nim-sets $\mathcal{N}(n, 3, \{1, 6\})$ for $1 \leq n \leq 14$:

$$\{\}, \{\}, \{0\}, \{1\}, \{0\}, \{1\}, \{0\}, \{1\}, \{0, 2\}, \{1, 3\}, \{0, 2\}, \{1, 3\}, \{0, 2\}, \{1, 3\}$$

and $\mathcal{N}(n, 3, \{1, 10\})$ for $1 \leq n \leq 22$:

$$\{\}, \{\}, \{0\}, \{1\}, \{0\}, \{1\}, \{0\}, \{1\}, \{0\}, \{1\}, \{0\}, \{1\}, \{0, 2\}, \{1, 3\}, \{0, 2\}, \{1, 3\}, \{0, 2\}, \{1, 3\}, \{0, 2\}, \{1, 3\}, \{0, 2\}, \{1, 3\}$$

That the first 2 (i.e., $p - 1$) terms are empty makes sense, since we cannot split a pile of size $< p$ into p piles. After that, we have the alternating subsequences that we saw earlier, in blocks of length $2c$. However, there is a bit more subtlety elsewhere. Let us compare $\mathcal{N}(n, 3, \{1, 6\})$ for $15 \leq n \leq 20$:

$$\{0, 1, 2\}, \{0, 1, 3, 4\}, \{0, 1, 2, 5\}, \{0, 1, 3, 4\}, \{0, 1, 2, 5\}, \{0, 1, 3, 4\}$$

with $\mathcal{N}(n, 3, \{1, 10\})$ for $23 \leq n \leq 32$:

$$\{0, 1, 2\}, \{0, 1, 3, 4\}, \{0, 1, 2, 5\}, \{0, 1, 3, 4\}, \{0, 1, 2, 5\}, \{0, 1, 3, 4\}, \{0, 1, 2, 5\}, \{0, 1, 3, 4\}, \{0, 1, 2, 5\}, \{0, 1, 3, 4\}$$

Here we see that the terms in position $2cq + 1 + (p - 1)$ might be irregular. Next compare $\mathcal{N}(n, 2, \{1, 6\})$ for $21 \leq n \leq 26$:

$$\{1, 2, 4, 6\}, \{0, 3, 5, 7\}, \{1, 2, 4, 6\}, \{0, 3, 5, 7\}, \{2, 6\}, \{0, 1, 3, 4, 7\}$$

with $\mathcal{N}(n, 2, \{1, 10\})$ for $33 \leq n \leq 42$:

$$\{1, 2, 4, 6\}, \{0, 3, 5, 7\}, \{1, 2, 4, 6\}, \{0, 3, 5, 7\}, \{1, 2, 4, 6\}, \{0, 3, 5, 7\}, \{2, 6\}, \{0, 1, 3, 4, 7\}$$

Here we see that the terms in positions $2cq - 1 + (p - 1)$ and $2cq + (p - 1)$ might be irregular. Putting this all together, we define our correspondence as follows:

Definition 14.

(a) Given $1 \leq r \leq 2c$, define

$$r' = \begin{cases} 1 & \text{if } r = 1 \\ 2 & \text{if } r = 2 \\ 3 & \text{if } 2 < r < 2c - 1 \text{ and } r \text{ is odd} \\ 4 & \text{if } 2 < r < 2c - 1 \text{ and } r \text{ is even} \\ 5 & \text{if } r = 2c - 1 \\ 6 & \text{if } r = 2c \end{cases}$$

In other words,

r	1	2	3	4	5	6	7	...	$2c - 3$	$2c - 2$	$2c - 1$	$2c$
r'	1	2	3	4	3	4	3	...	3	4	5	6

(b) Next suppose $p \geq 1$, $c \geq 3$, and $n \geq p$. Write $n = k + p - 1 = 2cq + r + p - 1$ with $1 \leq r \leq 2c$. Then for each $p \geq 1$, $n \geq p$, set

$$\phi_p(n) = \phi_p(2cq + r + p - 1) = 6q + r' + p - 1.$$

Since [3] proved [Theorem 3](#) for $c = 2, 3$, the following theorem finishes the proof of [Theorem 3](#).

Theorem 15. For $c \geq 3$

$$\mathcal{G}_{\{1, 2c\}}(n) = \mathcal{G}_{\{1, 6\}}(\phi_1(n)).$$

As we said above, we first prove that the ϕ -correspondence holds on the corresponding nim-sets. This is quite surprising, as it is true for all p , not just $p = 2$ and $p = 2c + 1$.

Lemma 16. For $c \geq 3$ and $k \geq 1$, $p \geq 2$,

$$\mathcal{N}(k + p - 1, p, \{1, 2c\}) = \mathcal{N}(\phi_p(k + p - 1), p, \{1, 6\}).$$

Table 5
The effect of ϕ_2 on remainders mod $2c$.

$s = r + 1$	1	2	3	4	5	6	7	...	$2c - 3$	$2c - 2$	$2c - 1$	$2c$
r	$2c$	1	2	3	4	5	6	...	$2c - 4$	$2c - 3$	$2c - 2$	$2c - 1$
r'	6	1	2	3	4	3	4	...	4	3	4	5
$r' + 1$	1	2	3	4	5	4	5	...	5	4	5	6

Table 6
The effect of ϕ_3 on remainders mod $2c$.

$s = r + 2$	1	2	3	4	5	6	7	...	$2c - 3$	$2c - 2$	$2c - 1$	$2c$
r	$2c - 1$	$2c$	1	2	3	4	5	...	$2c - 5$	$2c - 4$	$2c - 3$	$2c - 2$
r'	5	6	3	4	3	4	3	...	3	4	3	4
$r' + 2$	1	2	3	4	5	6	5	...	5	6	5	6

Table 7
The effect of ϕ_4 on remainders mod $2c$.

$s = r + 3$	1	2	3	4	5	6	7	...	$2c - 2$	$2c - 1$	$2c$
r	$2c - 2$	$2c - 1$	$2c$	1	2	3	4	...	$2c - 5$	$2c - 4$	$2c - 3$
r'	4	5	6	1	2	3	4	...	3	4	3
$r' + 3$	1	2	3	4	5	6	7	...	6	7	6

For example, the initial values of the sequence $\mathcal{N}(n, 3, \{1, 6\})$ are

- {}, {}.
- {0}, {1}, {0}, {1}, {0}, {1},
- {0, 2}, {1, 3}, {0, 2}, {1, 3}, {0, 2}, {1, 3},
- {0, 1, 2}, {0, 1, 3, 4}, {0, 1, 2, 5}, {0, 1, 3, 4},

and the initial values of the sequence $\mathcal{N}(n, 3, \{1, 8\})$ are

- {}, {}.
- {0}, {1}, {0}, {1}, {0}, {1}, {0}, {1},
- {0, 2}, {1, 3}, {0, 2}, {1, 3}, {0, 2}, {1, 3},
- {0, 1, 2}, {0, 1, 3, 4}, {0, 1, 2, 5}, {0, 1, 3, 4}, {0, 1, 2, 5}, {0, 1, 3, 4}

Before we prove this, we need some facts about ϕ for our base cases.

Lemma 17. Fix $c \geq 3$ and $C = \{1, 2c\}$.

- (a) For all p , ϕ_p preserves parity.
- (b) The map ϕ_1 induces bijections from the innumbers (outnumbers) of $C = \{1, 2c\}$ to the innumbers (outnumbers) of $C = \{1, 6\}$.
- (c) Suppose $2 \leq p \leq 4$ and h_1, \dots, h_p are all innumbers or all outnumbers. If it is not the case that $h_1 \equiv h_2 \equiv h_3 \equiv h_4 \equiv 2 \pmod{2c}$, then

$$\phi_1(h_1) + \phi_1(h_2) + \dots + \phi_1(h_p) = \phi_p(h_1 + h_2 + \dots + h_p).$$

- (d) Suppose O_n is a partition on n with p parts, and that $\lfloor O_n \rfloor$ and $\lceil O_n \rceil$ are partitions of a and b , respectively. If $1 \leq p \leq 3$, then $\phi_p(a) \leq \phi_p(n) \leq \phi_p(b)$.

Proof. To prove (a), note that the map $r \mapsto r'$ preserves parity.

Part (b) is clear from the construction of ϕ_1 .

For (c) and (d), we first examine the effects of ϕ_2, ϕ_3 , and ϕ_4 on the remainders of a number mod $2c$. The full calculations are in Tables 5, 6, and 7. Recall that in the definition of ϕ_p , the remainder of n mod $2c$ is $s = r + p - 1$. The tables all assume that $c \geq 4$, since if $c = 3$, then all maps ϕ_p are the identity, and the lemma holds trivially.

For (c), suppose that $p = 2$ and $h_1 = 2cq_1 + s_1$ and $h_2 = 2cq_2 + s_2$ are innumbers. Then $s_i = 1$ or 2 . Thus $s_1 + s_2 = 2, 3$, or 4 . Thus Table 5 implies that

$$\begin{aligned} \phi_2(h_1 + h_2) &= \phi_2(2c(q_1 + q_2) + s_1 + s_2) = 6(q_1 + q_2) + s_1 + s_2 \\ &= (6q_1 + s_1) + (6q_2 + s_2) = \phi_1(h_1) + \phi_1(h_2). \end{aligned}$$

The innumber cases for $p = 3$ and $p = 4$ are similar, except when $h_1 \equiv h_2 \equiv h_3 \equiv h_4 \equiv 2 \pmod{2c}$. In this case, $h_1 + h_2 + h_3 + h_4 \equiv 8 \pmod{2c}$, but $\phi_4(8) \neq 8$, so the equation fails. Since outnumbers must be either $2c$ or $1 \pmod{2c}$, an examination of the tables shows that the equation holds in these cases as well. For example, suppose that $p = 2$ and $h_1 = 2cq_1 + 2c$ and $h_2 = 2cq_2 + s_2$ are outnumberers, in particular $s_2 = 1$ or $2c$. Then we have

$$\begin{aligned} \phi_2(h_1 + h_2) &= \phi_2(2c(q_1 + q_2 + 1) + s_2) = 6(q_1 + q_2 + 1) + s'_2 \\ &= (6q_1 + 6) + (6q_2 + s'_2) = \phi_1(h_1) + \phi_1(h_2). \end{aligned}$$

Part (d) is immediate for $p = 1$, so let $p = 2$ and suppose $O_n = \{2cq_1 + s_1, 2cq_2 + s_2\}$. Then $\lfloor 2cq_i + s_i \rfloor = 2cq_i + a_i$, where $a_i = 1$ or 2 , and $\lceil 2cq_i + s_i \rceil = 2cq_i + b_i$, where $b_i = 1$ or $2c$. Then

$$\begin{aligned} \phi_2(2cq_1 + a_1 + 2cq_2 + a_2) &= 6q_1 + 6q_2 + a'_1 + a'_2 \\ &\leq \phi_2(6q_1 + 6q_2 + s_1 + s_2) \\ &\leq 6q_1 + 6q_2 + b'_1 + b'_2 \\ &= \phi_2(6q_1 + b_1 + 6q_2 + b_2) \end{aligned}$$

The equalities are from part (c). The inequalities are ensured by the values in Table 5. For example since $a_1 + a_2 \leq 4$, and so $a'_1 + a'_2$ is less than or equal to all other values in Table 5 to the right of it.

The case for $p = 3$ is similar, except that it is not the case $a'_1 + a'_2 + a'_3 \leq s_1 + s_2 + s_3$ when $a'_1 = a'_2 = a'_3 = 2$ and $s_1 + s_2 + s_3$ is odd and less than $2c$. In this case, $s_1 + s_2 + s_3 = 5$. However, this case never occurs, due to the last clause of Definition 8. \square

To explain the role of Lemma 17(c) in the proof of Lemma 16, let us look at an example with $p = 2$. Suppose $O_n = (2ac + 1, 4c + 2)$, where $n = 2ac + 1 + 4c + 2 = 2c(a + 2) + 3$. Then

$$\begin{aligned} \mathcal{G}_{\{1,2c\}}(O_n) &= \mathcal{G}_{\{1,2c\}}(2ac + 1) \oplus \mathcal{G}_{\{1,2c\}}(4c + 2) \\ &= \mathcal{G}_{\{1,6\}}(6a + 1) \oplus \mathcal{G}_{\{1,6\}}(12 + 2) = \mathcal{G}_{\{1,6\}}(O'_n). \end{aligned}$$

The first and last equalities are by definition, and the middle one will be by assuming the induction hypothesis on n from Theorem 15. Lemma 17(c) then verifies that the resulting option O'_n corresponds to O_n via ϕ_2 . In other words, that the third equality of the following holds:

$$n' = (6a + 1) + (12 + 2) = \phi_1(2ac + 1) + \phi_1(4c + 2) = \phi_2(2ac + 1 + 4c + 2) = \phi_2(n)$$

Proof of Lemma 16. Let $n = k + p - 1$. As we saw in our examples above, this holds trivially if $n < p$. Suppose $n \geq p$, and toward an induction that for all $n' < n$, both Lemma 16 and Theorem 15 hold.

Case 1: $2 \leq p \leq 3$ In order to show $\mathcal{N}(n, p, \{1, 2c\}) \subseteq \mathcal{N}(\phi_p(n), p, \{1, 6\})$, let $O_n = (h_1, \dots, h_p)$ be a partition of n with p parts. If O_n is either an entering or exiting partition, then

$$\begin{aligned} \mathcal{G}_{\{1,2c\}}(O_n) &= \mathcal{G}_{\{1,2c\}}(h_1) \oplus \dots \oplus \mathcal{G}_{\{1,2c\}}(h_p) && \text{Definition 2} \\ &= \mathcal{G}_{\{1,6\}}(\phi_1(h_1)) \oplus \dots \oplus \mathcal{G}_{\{1,6\}}(\phi_1(h_p)) && \text{induction on } n \\ &= \mathcal{G}_{\{1,6\}}(\phi_1(h_1), \dots, \phi_1(h_p)) && \text{Definition 2} \\ &\in \mathcal{N}(\phi_1(h_1) + \dots + \phi_1(h_p), p, \{1, 6\}) && \text{Definition 5} \\ &= \mathcal{N}(\phi_p(h_1 + \dots + h_p), p, \{1, 6\}) && \text{Lemma 17(c)} \\ &= \mathcal{N}(\phi_p(n), p, \{1, 6\}) \end{aligned}$$

If O_n is an intermediate partition, then consider its floor $\lfloor O_n \rfloor$ and ceiling $\lceil O_n \rceil$, partitions on a, b respectively, with $a < n < b$. By the above calculation, $\mathcal{G}_{\{1,2c\}}(\lfloor O_n \rfloor) \in \mathcal{N}(\phi_p(a), p, \{1, 6\})$, $\mathcal{G}_{\{1,2c\}}(\lceil O_n \rceil) \in \mathcal{N}(\phi_p(b), p, \{1, 6\})$, and $\mathcal{G}_{\{1,2c\}}(O_n) \in \mathcal{N}(\phi_1(h_1) + \dots + \phi_1(h_p), p, \{1, 6\})$. Furthermore,

$$\begin{aligned} \phi_p(a) &= \phi_1(\lfloor h_1 \rfloor) + \dots + \phi_1(\lfloor h_p \rfloor) \leq \phi_1(h_1) + \dots + \phi_1(h_p) \\ &\leq \phi_1(\lceil h_1 \rceil) + \dots + \phi_1(\lceil h_p \rceil) = \phi_p(b) \end{aligned}$$

where the inequalities are Lemma 17(d) and the equalities are Lemma 17(c). Let Q be a partition of $\phi_1(h_1) + \dots + \phi_1(h_p)$ with p parts satisfying $\mathcal{G}_{\{1,6\}}(Q) = \mathcal{G}_{\{1,2c\}}(O_n)$. By Lemma 17(a), $\phi_p(n)$ has the same parity as $\phi_1(h_1) + \dots + \phi_1(h_p)$, and Lemma 17(d) implies $\phi_p(a) \leq \phi_p(n) \leq \phi_p(b)$, so by Corollary 9, there is a partition R of $\phi_p(n)$ with p parts satisfying $\mathcal{G}_{\{1,6\}}(R) = \mathcal{G}_{\{1,2c\}}(O_n) \in \mathcal{N}(\phi_p(n), p, \{1, 6\})$.

We still need to show $\mathcal{N}(\phi_p(n), p, \{1, 6\}) \subseteq \mathcal{N}(n, p, \{1, 2c\})$. First notice that the map ϕ_p is surjective on the set of all partitions. Thus we will be done if we can show $\phi_p(n) = \phi_p(n') \implies \mathcal{N}(n, p, \{1, 2c\}) = \mathcal{N}(n', p, \{1, 2c\})$. Examining Tables 5 and 6, we see that we only need to show the following.

Claim. If $p = 2$ and $n \geq 5 \pmod{2c}$, then $\mathcal{N}(n, p, \{1, 2c\}) = \mathcal{N}(n - 1, p, \{1, 2c\}) \oplus 1$. Also, if $p = 3$ and $n \geq 6 \pmod{2c}$, then $\mathcal{N}(n, p, \{1, 2c\}) = \mathcal{N}(n - 1, p, \{1, 2c\}) \oplus 1$.

Note that in almost each such case if O_n is a partition of n , then O_n has a part of at least size $3 \pmod{2c}$. Thus, we can apply [Corollary 9](#) to obtain a partition O_{n-1} of $n - 1$ where $\mathcal{G}_{1,2c}(O_n) = \mathcal{G}_{1,2c}(O_{n-1}) \oplus 1$. The exceptional case is if the remainders of O_n are precisely $\{2, 2, 2\}$. If one of these is not an innumber, then we can again apply [Lemma 7](#). Otherwise, suppose $O_n = (12cq_1 + 4c + 2, 12cq_2 + 4c + 2, 12cq_3 + 4c + 2)$. Then we can use $O_{n-1} = (12cq_1 + 4c + 1, 12cq_2 + 4c + 1, 12cq_3 + 4c + 3)$.

Case 2: $p = 4$ [Corollary 11](#) says that once either m or $m \oplus 1$ enters the sequence $\mathcal{N}(n, 4, \{1, 2c\})$ (or $\mathcal{N}(n, 4, \{1, 6\})$), then the alternating subsequence $(m, m \oplus 1)$ remains in the sequence. Note that the alternating subsequence has the potential to enter twice: first as m at an odd n or $m \oplus 1$ at an even n , and second as m at an even n or $m \oplus 1$ at an odd n . Thus, since ϕ_4 preserves parity, we just need to show that each nim value enters $\mathcal{N}(n, 4, \{1, 2c\})$ at $n = k$ if and only if it enters $\mathcal{N}(n, 4, \{1, 6\})$ at $n = \phi_4(k)$.

Next [Lemmas 12](#) and [7](#) imply that we only need to check the values of entering partitions with a part of size 1. [Lemma 17\(b\)](#) implies that ϕ_1 induces a bijection from entering partitions with a part of size 1 to entering partitions with a part of size 1.

So suppose that $O_n = \{h_1, h_2, h_3, h_4\}$ is a partition of n with a part of size 1, and that it is an entering partition with respect to $\mathcal{C} = \{1, 2c\}$.

$$\begin{aligned} \mathcal{G}_{\{1,2c\}}(O_n) &= \mathcal{G}_{\{1,2c\}}(h_1) \oplus \cdots \oplus \mathcal{G}_{\{1,2c\}}(h_4) && \text{Definition 2} \\ &= \mathcal{G}_{\{1,6\}}(\phi_1(h_1)) \oplus \cdots \oplus \mathcal{G}_{\{1,6\}}(\phi_1(h_4)) && \text{induction on } n \\ &= \mathcal{G}_{\{1,6\}}(\phi_1(h_1), \dots, \phi_1(h_4)) && \text{Definition 2} \end{aligned}$$

But then $\{\phi_1(h_1), \dots, \phi_1(h_4)\}$ is a partition with a part of size 1, and is entering with respect to $\mathcal{C} = \{1, 6\}$. Then [Lemma 17\(c\)](#) says that it is a partition of $\phi_4(n)$.

Case 3: $p \geq 5$ We proceed by induction on p , using $p = 4$ as the base case. Note that [Lemma 17\(c\)](#) fails for $p \geq 5$, so the above argument does not apply. Also the following argument would not work above, since [Theorem 13](#) only holds if $p \geq 4$. Writing $n = k + p - 1 = 2cq + r + p - 1$, we have

$$\begin{aligned} \mathcal{N}(2cq + r + p - 1, p, \{1, 2c\}) &&& \\ = \mathcal{N}(2cq + r + (p - 1) - 1, p - 1, \{1, 2c\}) &&& \text{Theorem 13} \\ = \mathcal{N}(\phi_{p-1}(2cq + r + (p - 1) - 1), p - 1, \{1, 6\}) &&& \text{induction on } p \\ = \mathcal{N}(6q + r' + (p - 1) - 1, p - 1, \{1, 6\}) &&& \text{definition of } \phi_{p-1} \\ = \mathcal{N}(6q + r' + p - 1, p, \{1, 6\}) &&& \text{Theorem 13} \\ = \mathcal{N}(\phi_{p-1}(2cq + r + p - 1), p, \{1, 6\}) &&& \text{definition of } \phi_{p-1} \quad \square \end{aligned}$$

Now we can finally prove our main result:

Proof of Theorem 15. As in earlier proofs, we note that [Proposition 4](#) allows us to restrict our attention to $1 \leq n \leq 12c$.

Claim.

$$\begin{aligned} \text{(a)} \quad \mathcal{N}(n, 2c + 1, \{1, 2c\}) &= \mathcal{N}(\phi_1(n), 7, \{1, 6\}) \\ \text{(b)} \quad \mathcal{N}(n, 2, \{1, 2c\}) &= \begin{cases} \mathcal{N}(\phi_1(n), 2, \{1, 6\}) \cup \{1\} & \text{if } n = 8c + a, \\ & \text{for some } a \in \{5, 7, \dots, 2c - 3\} \\ \mathcal{N}(\phi_1(n), 2, \{1, 6\}) & \text{otherwise} \end{cases} \end{aligned}$$

Assume the claim. If it is not the case that $n = 8c + a$ for some $a \in \{5, 7, \dots, 2c - 3\}$, then it is clear that $\mathcal{G}_{\{1,2c\}}(n) = \mathcal{G}_{\{1,6\}}(\phi_1(n))$.

So suppose $a \in \{5, 7, \dots, 2c - 3\}$ and let $n = 8c + a$. Then we can see that

$$1 \in \mathcal{N}(\phi_1(8c + a), 7, \{1, 6\}) = \mathcal{N}(27, 7, \{1, 6\})$$

by considering, for example, the option $O_{27} = (13, 3, 3, 2, 2, 2)$. That is $\mathcal{G}_{\{1,6\}}((13, 3, 3, 2, 2, 2)) = 1$. Thus we conclude that

$$\begin{aligned} \mathcal{G}_{\{1,2c\}}(n) &= \text{mex} \{ \mathcal{N}(n, 2, \{1, 2c\}) \cup \mathcal{N}(n, 2c + 1, \{1, 2c\}) \} \\ &= \text{mex} \{ \mathcal{N}(\phi_1(n), 2, \{1, 6\}) \cup \{1\} \cup \mathcal{N}(\phi_1(n), 7, \{1, 6\}) \} \\ &= \text{mex} \{ \mathcal{N}(\phi_1(n), 2, \{1, 6\}) \cup \mathcal{N}(\phi_1(n), 7, \{1, 6\}) \} \\ &= \mathcal{G}_{\{1,6\}}(\phi_1(n)) \end{aligned}$$

Proof of claim. To prove (a), first note that if $n < 2c + 1$, then $\phi_1(n) < 7$, so we have $\mathcal{N}(n, 2c + 1, \{1, 2c\}) = \{ \} = \mathcal{N}(\phi_1(n), 7, \{1, 6\})$. So suppose $n = 2cq + 2c + r$ with $1 \leq r \leq 2c$ and $0 \leq q \leq 4$. Then

$$\begin{aligned} \mathcal{N}(n, 2c + 1, \{1, 2c\}) &= \mathcal{N}(2cq + r + 2c, 2c + 1, \{1, 2c\}) \\ &= \mathcal{N}(2cq + r + 6, 7, \{1, 2c\}) && \text{Theorem 13} \\ &= \mathcal{N}(6q + r' + 6, 7, \{1, 6\}) && \text{Lemma 16} \\ &= \mathcal{N}(\phi_1(n), 7, \{1, 6\}) \end{aligned}$$

For (b), again $\mathcal{N}(1, 2, \{1, 2c\}) = \{ \} = \mathcal{N}(\phi_1(1), 2, \{1, 6\})$, so let $n = 2cq + r + 1$ with $1 \leq r \leq 2c$ and $0 \leq q \leq 5$. Then we have

$$\begin{aligned} \mathcal{N}(n, 2, \{1, 2c\}) &= \mathcal{N}(2cq + r + 1, 2, \{1, 2c\}) \\ &= \mathcal{N}(6q + r' + 1, 2, \{1, 6\}) && \text{Lemma 16} \end{aligned}$$

If we examine Table 5, we see that $6q + r' + 1 = \phi_1(n)$, except when $r + 1 \in \{5, 7, \dots, 2c - 3\}$. In these cases, $\phi_1(n) = 6q + 3$, but $6q + r' + 1 = 6q + 5$. As it happens, we can use the computer to compute $\mathcal{N}(6q + 3, 2, \{1, 6\})$ and $\mathcal{N}(6q + 5, 2, \{1, 6\})$ for all $0 \leq q \leq 5$. They are:

q	0	1	2	3	4	5
$\mathcal{N}(6q + 3, 2, \{1, 6\})$	{1}	{1, 3}	{0, 1, 3, 4}	{1, 2, 4, 6}	{0, 2, 5, 6}	{0, 1, 3, 5, 7}
$\mathcal{N}(6q + 5, 2, \{1, 6\})$	{1}	{1, 3}	{0, 1, 3, 4}	{1, 2, 4, 6}	{0, 1, 2, 5, 6}	{0, 1, 3, 5, 7}

We see that they are equal except that when $q = 4$, $\mathcal{N}(6q + 5, 2, \{1, 6\}) = \mathcal{N}(6q + 3, 2, \{1, 6\}) \cup \{1\}$, proving the claim. \square

6. An extension

Let $2 \leq c_1 < \dots < c_k < \dots$. In this section we prove that the game cut with the (possibly finite) cut-set $\{1, 2c_1, \dots, 2c_k, \dots\}$ has the same nim-sequence as the game cut with cut-set $\{1, 2c_1\}$. We begin by proving a correspondence between nim-sets of the game with cut-set $\{1, 2c_1\}$ that we will use a repeated version of.

Lemma 18. For $p \geq 4, c \geq 2$,

$$\mathcal{N}(n, p + 2, \{1, 2c\}) \subset \mathcal{N}(n, p, \{1, 2c\}).$$

Proof.

$$\begin{aligned} \mathcal{N}(n, p + 2, \{1, 2c\}) &= \mathcal{N}(n - 2, p, \{1, 2c\}) && \text{Theorem 13 twice} \\ &\subset \mathcal{N}(n, p, \{1, 2c\}) && \text{Corollary 11 twice } \square \end{aligned}$$

Theorem 19. Let $C_1 = \{1, 2c_1, \dots, 2c_k, \dots\}$, where $2 \leq c_1 < \dots < c_k < \dots$ (possibly finite) and let $C_2 = \{1, 2c_1\}$. Then for $n \geq 1$,

$$\mathcal{G}_{C_1}(n) = \mathcal{G}_{C_2}(n).$$

Proof. We proceed by induction on n . For the base case, it is clear that $\mathcal{G}_{C_1}(1) = \mathcal{G}_{C_2}(1) = 0$. For the induction step, we assume the statement is true for all $n' < n$. This means that for any $p > 1$, $\mathcal{N}(n, p, C_1) = \mathcal{N}(n, p, C_2)$. Finally, Lemma 18 implies that $\bigcup_{x=c_1}^{\infty} \mathcal{N}(n, 2x + 1, \{1, 2c_1\}) = \mathcal{N}(n, 2c_1 + 1, \{1, 2c_1\})$, so the last step below holds even if C_1 is infinite. Hence

$$\begin{aligned} \mathcal{G}_{C_1}(n) &= \text{mex} \left\{ \mathcal{N}(n, 2, C_1) \cup \bigcup_{2c_i \in C_1} \mathcal{N}(n, 2c_i + 1, C_1) \right\} \\ &= \text{mex} \left\{ \mathcal{N}(n, 2, C_2) \cup \bigcup_{2c_i \in C_1} \mathcal{N}(n, 2c_i + 1, C_2) \right\} \\ &= \text{mex} \{ \mathcal{N}(n, 2, C_2) \cup \mathcal{N}(n, 2c_1 + 1, C_2) \} && \text{Lemma 18} \\ &= \mathcal{G}_{C_2}(n) \quad \square \end{aligned}$$

7. What is left?

In this section, we categorize the families of cut sets for which the nim-sequence of CUT remains unknown. There are 4 such families. Let X to be a non-empty set of even numbers, each of which is at least 4. Let Y to be a non-empty set of odd numbers, each of which is at least 5. Let $x = 2c$ and y be the smallest elements of X and Y , respectively.

Family A: $C = \{1, 3\} \cup X$, or $C = \{1, 3\} \cup X \cup Y$

We already know from [3, Propositions 8] that the nim-sequence for $C = \{1, 3, 2c\}$ is $(0, 1)^c(+2)$.

Conjecture 1. *The nim-sequence for all games of CUT in this family are all precisely $(0, 1)^c(+2)$.*

Family B: $C = \{1\} \cup X \cup Y$

The nim-sequence of this family seems to have some resemblance to the nim-sequence for $C = \{1, 2c\}$ when $c \geq 2$, but we cannot make a full conjecture at this time. The following partial extension of Theorem 19 seems to be true.

Conjecture 2. *If $3x < y$, then $\mathcal{G}_C(n) = \mathcal{G}_{\{1,x\}}(n)$ for $n \geq 1$.*

We note that proving the arithmetic-periodicity of Families A and B would imply Conjecture 1 of [3].

Family C: $\{1, 2\} \subseteq C, 3 \notin C, C \neq \{1, 2\}$.

It is not so clear how to categorize the patterns of this family. However, we do observe:

Conjecture 3. *The nim-sequence for all games of CUT in Family C are all ultimately arithmetic-periodic.*

Family D: $C = \{1, 2\}$

The first 36 terms of the nim-sequence for this game of CUT are

0,	1,	2,	3,	1,	4,	3,	2,	4,	5,	6,	7,
8,	9,	7,	6,	9,	8,	11,	10,	12,	13,	10,	11,
13,	12,	15,	14,	16,	17,	5,	15,	17,	16,	19,	18

It is supposed that the nim-sequence for this particular version of CUT is the most difficult to analyze. We also cannot find any pattern here. In [3], it was shown that this game is equivalent to the take-and-break game with hexadecimal code 0.7F.

Data availability

The associated Maple program can be found at the second author’s website: www.thotsaporn.com.

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