Inversions and Major Index for Permutations

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1 Introduction

It is well known that there are $n!$ permutations of the set $\{1, 2, \ldots, n\}$. Stern [4] asked the question of how many inversions there are in these $n!$ permutations. An *inversion* occurs when a larger number appears before a smaller number. For example, the permutation 4132 has 4 inversions: 3 from 4 appearing before $1, 2, 3$ and 1 from 3 appearing before 2. Stern's question was answered by Terquem [5] who showed that the total number of inversions in the permutations of $\{1, 2, \ldots, n\}$ is

$$I(n) = n!n(n-1)/4. \tag{1}$$

Shortly after this, Rodrigues [3] gave two other solutions to Stern's problems. More importantly, he found the generating function for these inversions: If $I(n, k)$ denotes the number of permutations of $\{1, 2, \ldots, n\}$ with k inversions, then

$$\sum_{k \geq 0} I(n, k) q^k = 1(1+q) \cdots (1 + q + \cdots + q^{n-1}). \tag{2}$$

Rodrigues's work was ignored and then forgotten. MacMahon [2] studied a more general problem. He studied the inversion problem for multisets, i.e., he allowed the integers $1, \ldots, n$ to be repeated. In addition, he introduced a second statistic, which is now called the major index.

For permutation of $\{1, 2, \ldots, n\}$, a *descent* occurs when a larger number appears immediately before a smaller number. The *major index* of a permutation is the sum of the locations of the first number in each pair which is a descent. The example given before $\underline{4}1\underline{3}2$, has major index $1 + 3 = 4$, since a descent from 4 to 1 occurs in the first space and another from 3 to 2 in the third space. While the number of inversions and major index are the same for 4132, this is usually not true. The permutation $\underline{4}\underline{3}12$ has 5 inversions and its major index is $1 + 2 = 3$.

Here are the data for the permutations of $\{1, 2, 3\}$:

Permutation	**Inversions**	**Major Index**
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

Notice that these two statistics are equidistributed, i.e., the number of permutations with k inversions is equal to the number of permutations with major index equal to k. MacMahon proved this in the more general case of multisets by showing that both statistics have the same generating function.

Schutzenberger suggested privately to Foata that there should be a combinatorial way to showing these two statistics are equidistributed Foata [1] found a bijection which does this. However, Foata's bijection is not easily understood. In a combinatorics course at the University of Wisconsin, Madison, the professor, Richard Askey, asked if anyone could find a combinatorial argument which shows the equidistribution of these two statistics for the set $\{1, 2, \ldots, n\}$. My solution is given in Section 3. Section 2 gives the background information on inversions and permutations.

2 Permutations and inversions

Let $I(n)$ denote the number of inversions of the $n!$ permutations of $\{1, 2, \ldots, n\}$. There are a number of ways to find the value of $I(n)$. Rodrigues [3] pointed out that in each permutations $p_1, p_2, \ldots, p_n$, and associated permutation $p_n, p_{n-1}, \ldots, p_1$ the pair p_i, p_j is an inversion in one but not the other, so half of the total number of pairs in the $\frac{n!n(n-1)}{2}$ pairs in all permutations are inversions. Thus there are $\frac{n!n(n-1)}{4}$ inversions.

A second way, Terquem used a recurrence relations between $I(n)$ and $I(n+1)$ and show that $I(n) = \frac{n!n(n-1)}{4}$ satisfies this relation. Each permutation $p_1, p_2, \ldots, p_n$ of $\{1, 2, \ldots, n\}$ gives rise to $n+1$ permutations of $\{1, 2, \ldots, n, n+1\}$ by inserting $n+1$ before any of the p_i or after p_n. Then

$$I(n+1) = \underbrace{I(n)}_{n+1 \text{ last}} + \underbrace{(I(n)+n!)}_{\text{second to last}} + \underbrace{(I(n)+2n!)}_{\text{third to last}} + \cdots + \underbrace{(I(n)+nn!)}_{n+1 \text{ first}}, \qquad (3)$$

since insertion of $n+1$ before k adds $n+1-k$ inversions to each permutation. This equation is

$$\begin{aligned} I(n+1) &= (n+1)I(n) + \frac{n\ n(n+1)}{2} \qquad (4) \\ &= (n+1)I(n) + \frac{(n+1)!n}{2}. \end{aligned}$$

An easy calculation shows that

$$I(n) = \frac{n!n(n-1)}{4} \qquad (5)$$

satisfies (4).

Alternatively, it is possible to solve (4) using a method which is well known in elementary differential equations. First solve homogeneous equation

$$I(n+1) = (n+1)I(n)$$

to get

$$I(n) = c \cdot n!$$

Then use this variation of parameters to simplify (4) by setting

$$I(n) = J(n)n!,$$

substituting this in (4) we get

$$J(n+1) = J(n) + \frac{n}{2}.$$

And this implies

$$\begin{aligned} J(n) = J(1) &= \frac{1+2+\cdots+(n-1)}{2} \\ &= \frac{1}{2}\binom{n}{2} \end{aligned}$$

since $J(1) = 0$. This gives Terquem's formula.

The generating function found by Rodrigues is also easy to find. If

$$G_n(q) = \sum_{k \geq 0} I(n,k)q^k,$$

then the same insertion argument which proved (3) gives

$$G_{n+1}(q) = G_n(q)(1+q+\cdots+q^n). \tag{6}$$

Iteration of (6) and the case $n = 2$ gives Rodrigues's formula

$$\sum_{k \geq 0} I(n,k)q^k = 1(1+q)\cdots(1+q+\cdots+q^{n-1}). \tag{7}$$

Rodrigues first solved Stern's problem by differentiating (7) with respect to q and then setting $q = 1$. We leave this as an exercise for the reader.

3 Equidistribution of inversions and major index

We will give an induction proof that the number of inversions and the major index are equidistributed. First, a permutation of $\{1, 2, \ldots, n\}$ with k inversions has an easy pattern for the number of inversions when $n+1$ is inserted into this permutation. When placed after n, there are still k inversions. When $n+1$ is inserted before the j-th element from the right, there are $k+j$ inversions. Thus the number of inversions when inserting $n+1$ into all of the $n+1$ places in this permutation give permutations with $k, k+1, \ldots, k+n$ inversions.

The pattern when the major index is considered is not as transparent, but it gives rise to the same set of numbers. To see what happens, consider the following example, when 8 is inserted into the permutation 5724631 of $\{1, 2, \ldots, 7\}$

Permutation	**Major Index**
$5724631\underline{8}$	$2+5+6=13$
$572463\underline{8}1$	$2+5+7=14$
$57246\underline{8}31$	$2+6+7=15$
$5724\underline{8}631$	$2+5+6+7=20$
$572\underline{8}4631$	$2+4+6+7=19$
$57\underline{8}24631$	$3+6+7=16$
$5\underline{8}724631$	$2+3+6+7=18$
$\underline{8}5724631$	$1+3+6+7=17$

Notice that the major index numbers are $13, 14, 15, \ldots, 20$. The inversion numbers for this permutation are not the same, since there is no reason that the number of inversions end the major index of 5724637 are the same. However, if we can explain why the major index for all the insertions of $n+1$ give numbers which are $k, k+1, \ldots, k+n$ when the major index of the original permutation is k, we are finished.

We can categorize the insertion into 2 cases.

Case a) When we add the numbers $n+1$ between the descents in the permutation or at the end.

For example

Permutation	**Major index**
$5724631\underline{8}$	$2+5+6+=13$
$572463\underline{8}1$	$2+5+7+=14$
$57246\underline{8}31$	$2+6+7+=15$
$57\underline{8}24631$	$3+6+7+=16$
$\underline{8}5724631$	$1+3+6+7+=17$

We can see that after inserting the number $n+1$ in each of the descents from right to left in the permutation, the major index increases one each time.

Case b) When we add the number $n+1$ between the ascents in the permutation. For example

Permutation	**Major index**
$5724\underline{8}631$	$2+5+6+7=20$
$572\underline{8}4631$	$2+4+6+7=19$
$5\underline{8}724631$	$2+3+6+7=18$
$\underline{8}5724631$	$1+3+6+7=17$

We can see that after inserting the number $n+1$ to each of the ascents from right to left in the permutation, the major index decreases one each time.

Proof Case (a) We show that the major index increases one each time we insert the number $n+1$ between the descent to the next descent to the left.

Proof. Consider

Permutation	**Major index**
$57246\underline{8}31$	$2+6+7=15$
$57\underline{8}24631$	$3+6+7=16$

as an example.

Two things happen when comparing the insertion of number $n+1$ in each permutation.

1) *At the new position of number* $n+1$. In the new permutation, the number $n+1$ adds a value to the major index, while the major index of the descent that number $n+1$ was inserted in is gone. So the major index increases by one. For example in $57\underline{8}24631$, we gained 3 from inserting 8 and lost 2 as the major index from 7 is gone. So the major index increases by one.

2) *At the original position of number* $n+1$. Although the major index from number $n+1$ in the original permutation is gone, we get the same major index after the inserted number $n+1$ is moved to the descent on the left.

For example in $57246\underline{8}31$ there is a major index of 6 from number 8 which is lost. But in $57\underline{8}24631$ there is a gain of 6 in the major index from number 6.

So from 1) and 2) show that the major index increases one each time we move the number $n+1$ in a descent to the next descent on the left.

Proof Case b). We show that the major index decreases by one each time we insert the number $n+1$ between ascent relative to inserting it in the next ascent on the left.

Proof. Consider

Permutation	**Major index**
$572\underline{8}4631$	$2+4+6+7=19$
$5\underline{8}724631$	$2+3+6+7=18$

Two things happen when comparing an insertion of number $n+1$ in each permutation

1) Number $n+1$ moves $\underline{a}$ position(s) to the left causing the major index to decrease by a. In the example number 8 moves 2 positions to the left causing the major index to decrease by 2.

2) The number $n+1$ will pass $(a-1)$ descents. So it will push each of these $(a-1)$ descents one position to the right, so the major index increases by $(a-1)$. In the example there is 1 decreasing sequence "72" that is pushed to the right. So from 1) and 2) the major index changed $= -a+(a-1) = -1$. So the major index decreases one each time we insert the number $n+1$ between an ascent and the next increasing sequence on the left.

Conclusion

Let m be a number of descents in the original permutation.

Case a) the major index increases from its original value by $0, 1, \ldots, m+1$.

Case b) the major index decreases from its original value by $n, n-1, \ldots, m+1$.

(*Note* that after adding $n+1$ to the first ascent on the right we gain n to the major index. Since we gain ℓ from insert number $n+1$ at position ℓ and gain $n-\ell$ as there are $(n-1)$ descents that are pushed by number $n+1$ to the right.)

So we gain $0, 1, 2, \ldots, n$ of the major index after inserting the number $(n+1)$ to all positions in any permutation of $\{1, 2, \ldots, n\}$.

The $n = 1$ case is trivial, since there are no inversions and the major index is zero. This argument proves the major index and the number of inversions are equidistributed, since it is the n to $n+1$ step of an induction argument.

References

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