Rado Numbers of Regular Non-homogeneous Equations

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Ramsey Theory

"Every system possesses a large subsystem with a higher degree of organization than the original system." –Burkill and Mirsky

Ramsey theory is Paul Erdős first love and remains his love through out his career. It is also the central interest of the branch of mathematic called *Hungarian mathematics*.

Paul Erdős

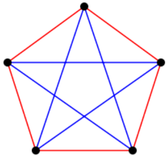
Ramsey Numbers for Graph

Theorem (Ramsey's Theorem for Two Colors, 1930)

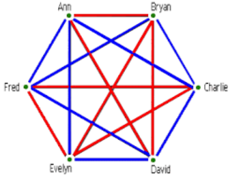
Let's $k, l \geq 2$. There exists a least positive integer $R = R(k, l)$ such that for every edge-coloring of K_R, with colors red or blue, admits either a red K_k subgraph or a blue K_l subgraph.

Example:
$R(3, 3) = 6, R(4, 4) = 18, R(5, 5) \in [43, 49], R(6, 6) \in [102, 165].$

$R(3,3) = 6$



No monochromatic triangle



Must have a monochromatic triangle

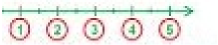
Schur Numbers

Theorem (Issai Schur, 1916)

For any t colors, $t \geq 1$, there is a number $s(t)$ such that for any t-coloring on the interval $[1, s(t)]$, there must be a monochromatic solution to $x + y = z$ where x, y and z are the positions on the interval.

Example: $s(1) = 2, s(2) = 5, s(3) = 14, s(4) = 45, s(5) = 161.$

$s(2) = 5$



Positive integer line

$$1 \to \text{Red}$$
$$2 \to \text{Blue} \qquad (1+1=2)$$
$$4 \to \text{Red} \qquad (2+2=4)$$
$$3 \to \text{Blue} \qquad (1+3=4)$$
$$5 \to ??$$

Rado Numbers

Later on, Richard Rado, a Ph.D. student of Schur, generalized Schur's work to a linear homogeneous equation $\mathcal{E} : \sum_{i=1}^{k} c_i x_i = 0$ and find the condition for regularity of these equations. The numbers $r(\mathcal{E}; t)$, analog to Schur numbers, are called Rado numbers.

Note that we call the equation *regular* if $r(\mathcal{E}; t)$ exists for any given t colors.

Rado's Theorem

Theorem (Rado's Single Equation Theorem)

Let $k \geq 2$. Let $c_i \in \mathbb{Z} - \{0\}, 1 \leq i \leq k$, be constants. Then

$$\sum_{i=1}^{k} c_i x_i = 0$$

is regular if and only if there exists a nonempty set $D \subseteq \{c_i, \ 1 \leq i \leq k\}$ such that $\sum_{d \in D} d = 0$.

Example: Let $a, b \geq 1$.

$$\mathcal{E} : ax + by - bz = 0$$

is regular, i.e.

$$r(ax + y - z = 0; 2) = a^2 + 3a + 1.$$

Rado's Numbers for Non-homogeneous Equations

Some results on non-homogeneous equations.

Theorem (Schaal (1995), [2])

For $b \geq 1$,

$$r(x+y-z=b;2) = b - \left\lceil \frac{b}{5} \right\rceil + 1,$$

$$r(x+y-z=-b;3) = 13b+14,$$

$$r(x+y-z=b;3) = b - \left\lceil \frac{b}{14} \right\rceil + 1.$$

Regularity Condition for a Non-homogeneous Equation

An analog to Rado's Theorem which gives the regularity condition for a linear non-homogeneous equation is given below.

Theorem

Let $k \geq 2$ and let $b, c_1, c_2, \ldots, c_k$ be nonzero integers. Let $\mathcal{E}(b)$ be the equation

$$\sum_{i=1}^{k} c_i x_i = b,$$

and let $s = \sum_{i=1}^{k} c_i$. Then $\mathcal{E}(b)$ is regular if and only if one of the following conditions holds:

(i) $\dfrac{b}{s} \in \mathbb{Z}^+$;

(ii) $\dfrac{b}{s}$ *is a negative integer and $\mathcal{E}(0)$ is regular.*

Main Results

We partially quantify Theorem above.
We consider Rado numbers of the regular equation $\mathcal{E}(b)$ of the form

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k + b,$$

where $c_i \in \mathbb{Z}^+$ for all i. We give the upper bounds and the sufficient condition for the lower bounds for t-colors Rado numbers $r(\mathcal{E}(b); t)$ in term of $R_c(t) := r(\mathcal{E}(0); t)$ for $b > 0$ and $b < 0$.

Main Results; case $\tilde{b} < 0$, Upper Bounds

Theorem

Consider equation $\mathcal{E}(\tilde{b}) = \mathcal{E}(-b)$ of the form

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k - b, \qquad c_i > 0,\ b > 0.$$

Let $s = \sum_{i=1}^{k-1} c_i - 1$. If $s|b$ and $\mathcal{E}(0)$ is t-regular then

$$r(\mathcal{E}(-b); t) \le \left(\frac{b}{s} + 1\right) \cdot R_C(t) - \frac{b}{s}.$$

Main Results; case $\tilde{b} < 0$, Lower Bounds

Next we define a sufficient condition for the lower bounds.

Definition (excellence condition)

The coloring on an interval $[1, n]$ satisfies an excellence condition if it does not contain any mono. solution to

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} + j = x_k,$$

for each j, $0 \leq j \leq s = \sum_{i=1}^{k-1} c_i - 1$.

Example (Schur Equation)

[Red,Blue,Blue,Red] satisfies the excellence condition for $x + y = z$ i.e no mono. solution for $x + y = z$ and $x + y + 1 = z$.

Main Results; case $\tilde{b} < 0$, Lower Bounds

Theorem

Consider the equation $\mathcal{E}(\tilde{b}) = \mathcal{E}(-b)$ of the form

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k - b, \qquad \text{where } c_i > 0,\ b > 0.$$

Let $s = \sum_{i=1}^{k-1} c_i - 1$. If $s|b$ and there is a coloring on the interval $[1, n]$ which satisfies an excellence condition then

$$r(\mathcal{E}(-b); t) \geq \left(\frac{b}{s} + 1\right) \cdot n + 1.$$

Main Results; case $\tilde{b} < 0$

We note that the upper bounds and lower bounds meet if there is a good coloring of length $n = R_C(t) - 1$ that satisfies the excellence condition.

Corollary

For $m > 0$,

$$
\begin{aligned}
r(x + y - z = -m; 3) &= 13m + 14, \\
r(x + y + z - w = -2m; 3) &= 42m + 43, \\
r(x_1 + x_2 + x_3 + x_4 - x_5 = -3m; 3) &= 93m + 94, \\
r(x_1 + x_2 + x_3 + x_4 + x_5 - x_6 = -4m; 3) &= 172m + 173.
\end{aligned}
$$

Main Results; case $\tilde{b} > 0$, Upper Bounds

Similar to the negative $\tilde{b}$ case.

Theorem

Consider the equation $\mathcal{E}(\tilde{b}) = \mathcal{E}(b)$ *of the form*

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k + b, \qquad \textit{where } c_i > 0, \quad b > 0.$$

Let $s = \sum_{i=1}^{k-1} c_i - 1$. *If* $s|b$ *and* $\mathcal{E}(0)$ *is t-regular then*

$$r(\mathcal{E}(b); t) \leq \frac{b}{s} - \left\lceil \frac{b}{s \cdot R_C(t)} \right\rceil + 1.$$

Main Results; case $\tilde{b} > 0$, Lower Bounds

Theorem

Consider the equation $\mathcal{E}(\tilde{b}) = \mathcal{E}(b)$ of the form

$$c_1 x_1 + c_2 x_2 + \cdots + c_{k-1} x_{k-1} = x_k + b, \qquad \text{where } c_i > 0, \quad b > 0.$$

Let $s = \sum_{i=1}^{k-1} c_i - 1$. If $s|b$ and there is a coloring on the interval $[1, n]$ which satisfies the excellence condition then

$$r(\mathcal{E}(b); k) \geq \frac{b}{s} - \left\lceil \frac{b}{s \cdot (n+1)} \right\rceil + 1.$$

Final Remark

So far, our results were obtained by checking the excellence condition of each good coloring. For the 2-coloring and 3-coloring, it seems that there are always colorings of length $n = R_C(t) - 1$ to the equations

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k, \qquad \text{where } c_i > 0$$

that satisfy the excellence condition. Thus it makes sense to make the following conjecture.

Conjecture

Conjecture

For $t = 2$ or 3, fix constants $c_1, c_2, \ldots, c_{k-1}$. Consider the equation $\mathcal{E}(\tilde{b})$ of the form

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k + \tilde{b}, \qquad \text{where } c_i > 0.$$

Let $s = \sum_{i=1}^{k-1} c_i - 1$. If $s|\tilde{b}$ and $\mathcal{E}(0)$ is t-regular then

$$r(\mathcal{E}(\tilde{b}); t) = \begin{cases} \dfrac{\tilde{b}}{s} - \left\lceil \dfrac{\tilde{b}}{s \cdot R_C(t)} \right\rceil + 1, & \text{for } \tilde{b} > 0, \\ -\dfrac{\tilde{b}}{s} \cdot (R_C(t) - 1) + R_C(t), & \text{for } \tilde{b} < 0. \end{cases}$$

References

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