Rado Numbers of Regular Nonhomogeneous Equations

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August 28, 2019

Abstract

We consider Rado numbers of the regular equations $\mathcal{E}(b)$ of the form

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k + b,$$

where $b \in \mathbb{Z}$ and $c_i \in \mathbb{Z}^+$ for all i. We give the upper bounds and the sufficient condition for the lower bounds for t-color Rado numbers $r(\mathcal{E}(b);t)$ in terms of $r(\mathcal{E}(0);t)$ for both $b > 0$ and $b < 0$. We also give examples where the exact values of Rado numbers are obtained from these results.

1 Introduction

In 1916 Issai Schur [8] showed that for any t colors, $t \geq 1$, there is a least positive integer $s(t)$ such that for any t-coloring on the interval $[1, s(t)]$, there must be a monochromatic solution to $x + y = z$ where x, y and z are positions on the interval. This result is part of Ramsey Theory. The numbers $s(t)$ are called *Schur numbers*. For example $s(2) = 5$ and the longest possible interval that avoids the mono solution to $x + y = z$ is $[1, 2, 2, 1]$ (1 represents red color and 2 represents blue color, for example). For 3 colors, $s(3) = 14$ and one of the longest interval that avoids the mono solution to $x + y = z$ is $[1, 2, 2, 1, 3, 3, 3, 3, 3, 1, 2, 2, 1]$. It is also known that $s(4) = 45$ and $s(5) = 161$. We call the equation *t-regular* if $s(t)$ exists for a given t and *regular* if $s(t)$ exists for all t, $\; t \geq 1$.

Later on, Richard Rado, a Ph.D. student of Schur, generalized Schur's work to a linear homogeneous equation $\sum_{i=1}^{k} c_ix_i = 0$ and found the condition for regularity of these equations, [3, 4].

Theorem 1 (Rado's Single Equation Theorem)**.** *Let $k \geq 2$. Let $c_i \in \mathbb{Z} - \{0\}, 1 \leq i \leq k$, be constants. Then*

$$\sum_{i=1}^{k} c_i x_i = 0$$

is regular if and only if there exists a nonempty set $D \subseteq \{c_i,\ \ 1 \leq i \leq k\}$ such that $\sum_{d \in D} d = 0$.

As with Schur numbers, for a linear equation $\mathcal{E}$, we denote by $r(\mathcal{E}; t)$ the minimal integers, if it exists, such that any t-coloring of $[1, r(\mathcal{E}; t)]$ must admit a monochromatic solution to $\mathcal{E}$. The numbers $r(\mathcal{E}; t)$ are called *t-color Rado numbers for equation $\mathcal{E}$.*

An analog to Rado's Theorem which gives the regularity condition for a linear non-homogeneous equation is given below.

Theorem 2. *Let $k \geq 2$ and let $b, c_1, c_2, \ldots, c_k$ be nonzero integers. Let $\mathcal{E}(b)$ be the equation*

$$\sum_{i=1}^{k} c_i x_i = b,$$

and let $s = \sum_{i=1}^{k} c_i$. Then $\mathcal{E}(b)$ is regular if and only if one of the following conditions holds:

(i) $\dfrac{b}{s} \in \mathbb{Z}^+$;

(ii) $\dfrac{b}{s}$ *is a negative integer and $\mathcal{E}(0)$ is regular.*

We note that it is possible that an equation does not have a mono solution for a coloring on $\mathbb{Z}^+$. For example, the coloring $[1, 2, 1, 2, 1, 2, \ldots]$ avoids the mono solution to the equation $x + y = 2b + 1$ for any $b \geq 0$. Also some equations are t-regular but not regular. For example, $3x + y - z = 2$ is 2-regular with $r(\mathcal{E}; 2) = 8$ but not regular according to Theorem 2.

In this paper, we partially quantify Theorem 2 by giving Rado numbers to equations $\mathcal{E}(\tilde{b})$ of the form

$$c_1 x_1 + c_2 x_2 + \cdots + c_{k-1} x_{k-1} = x_k + \tilde{b}, \tag{1}$$

where $c_i \in \mathbb{Z}^+$ for all i and $\tilde{b}$ satisfies the condition (i) or (ii) of Theorem 2. The Rado numbers of (1) will be written in term of the Rado numbers of the corresponding homogeneous equation, $\mathcal{E}(0)$.

In order to distinguish the Rado numbers of the homogeneous equation from those of the non-homogeneous one, we denote by $R_C(t) = R_{[c_1, c_2, \ldots, c_{k-1}]}(t)$ the Rado number of the homogeneous equation, $\mathcal{E}(0)$, with t colors.

2 Main Results; case $\tilde{b} < 0$

We consider the Rado numbers of (1) where the constant $\tilde{b}$ is negative. Theorem 3 gives the upper bounds and Theorem 4 gives a sufficient condition for the lower bounds.

Theorem 3. *Consider equation $\mathcal{E}(\tilde{b}) = \mathcal{E}(-b)$ of the form*

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k - b, \qquad c_i > 0,\, b > 0.$$

Let $s = \sum_{i=1}^{k-1} c_i - 1$. If $s|b$ and $\mathcal{E}(0)$ is t-regular then

$$r(\mathcal{E}(-b); t) \leq \left(\frac{b}{s} + 1\right) \cdot R_C(t) - \frac{b}{s}.$$

Proof. Assume $s|b$ and $\mathcal{E}(0)$ is t-regular. Let $r = (\frac{b}{s} + 1) \cdot R_C(t) - \frac{b}{s}$.

We now show that there is no good coloring on the interval $[1, r]$.

Define an injective map f from $[1, R_C(t)]$ to $[1, r]$ by

$$f(w) = \left(\frac{b}{s} + 1\right) \cdot w - \frac{b}{s}.$$

Notice that the k-tuple $(w_1, w_2, \ldots, w_{k-1}, \sum_{i=1}^{k-1} c_iw_i)$ of the equation

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k$$

is made to correspond to the k-tuple $(f(w_1), f(w_2), \ldots, f(w_{k-1}), f(\sum_{i=1}^{k-1} c_iw_i))$ in

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k - b.$$

Please check it yourself!!!

Now given any coloring α on $[1, r]$, we define the coloring χ on the interval $[1, R_C(t)]$ by

$$\chi(w) := \alpha(f(w)), \qquad w = 1, 2, \ldots, R_C(t).$$

From the definition of the Rado number, any coloring on $[1, R_C(t)]$ must contain a mono tuple to $\mathcal{E}(0)$. Hence there is also a mono tuple on $[1, r]$ to $\mathcal{E}(-b)$. □

Next we define a sufficient condition for the lower bounds.

Definition (excellence condition)**.** The coloring on an interval $[1, n]$ satisfies an excellence condition if it does not contain any mono solution to

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} + j = x_k,$$

for each j, $0 \leq j \leq s = \sum_{i=1}^{k-1} c_i - 1$.

Theorem 4. *Consider the equation $\mathcal{E}(\tilde{b}) = \mathcal{E}(-b)$ of the form*

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k - b, \qquad \text{where } c_i > 0,\ b > 0. \tag{2}$$

Let $s = \sum_{i=1}^{k-1} c_i - 1$. If $s|b$ and there is a coloring on the interval $[1, n]$ which satisfies an excellence condition then

$$r(\mathcal{E}(-b); t) \geq \left(\frac{b}{s} + 1\right) \cdot n + 1.$$

Proof. Assume $s|b$ and let χ be the coloring on $[1, n]$ that satisfies an excellence condition to the equation

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} + j = x_k, \quad 0 \leq j \leq s.$$

Let $r = (\frac{b}{s} + 1) \cdot n + 1$.

We show that there is a "good coloring" to $\mathcal{E}(-b)$ on the interval $[1, r-1] = [1, \left(\frac{b}{s} + 1\right) \cdot n]$.

We define the coloring α on $[1, \left(\frac{b}{s} + 1\right) \cdot n]$ by

$$\alpha(i) = \chi\left(\left\lceil \frac{i}{\frac{b}{s} + 1} \right\rceil\right).$$

Basically, we create the coloring by repeating each point of the original coloring on the $[1, n]$ interval $\frac{b}{s} + 1$ times. We now prove the statement by contradiction:

Assume there is a mono k-tuple on $[1, \left(\frac{b}{s} + 1\right) \cdot n]$ to equation (2) written in the form

$$\left(d_1\left(\frac{b}{s}+1\right) - e_1, d_2\left(\frac{b}{s}+1\right) - e_2, \ldots, d_{k-1}\left(\frac{b}{s}+1\right) - e_{k-1}, \left(\frac{b}{s}+1\right) \cdot \sum_{i=1}^{k-1} c_i d_i - \sum_{i=1}^{k-1} c_i e_i + b\right),$$

where $1 \leq d_i \leq n$ for all i and $0 \leq e_i \leq b/s$.

Notice that $\alpha(d_i(\frac{b}{s} + 1) - e_i) = \chi(d_i)$. However, by this mapping, we have the mono k-tuple in χ as

$$\left(d_1, d_2, \ldots, d_{k-1}, \sum_{i=1}^{k-1} c_i d_i + \left\lceil \frac{b - \sum_{i=1}^{k-1} c_i e_i}{\frac{b}{s} + 1} \right\rceil\right)$$

But this is a mono solution to

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} + j = x_k,$$

for some j, $0 \leq j \leq \left\lceil \frac{sb}{b+s} \right\rceil$ which contradicts the excellence condition of χ we assumed it to have. □

We note that the upper bounds and lower bounds meet if there is a good coloring of length $n = R_C(t) - 1$ that satisfies the excellence condition.

Corollary 5. *Consider the equation $\mathcal{E}(-b)$,*

$$x_1 + x_2 + \cdots + x_{k-1} = x_k - b, \quad \text{with } k \geq 2, \ \ b > 0 \text{ and } (k-2)|b.$$

We let $m = b/(k-2)$. Then

$$r(\mathcal{E}(-b); 2) = (m+1)(k^2 - k - 2) + 1.$$

Proof. It is known (i.e. Theorem 8.23 of [2]) that

$$r(x_1 + x_2 + \cdots + x_{k-1} = x_k; 2) = k^2 - k - 1, \quad \text{for } k \geq 2.$$

The coloring $\chi = [1^{k-2}, 2^{(k-1)(k-2)}, 1^{k-2}]$ satisfies the excellence condition for each k. The result follows from Theorems 3 and 4. □

This result agrees with Theorems 9.14 and 9.26 of [2] which applies to any 2-coloring but for a more general b (not only $(k-2)|b$). However, our result applies to any t-coloring.

Corollary 6. *For $m > 0$,*

$$\begin{aligned}
r(x + y - z = -m; 3) &= 13m + 14, \\
r(x + y + z - w = -2m; 3) &= 42m + 43, \\
r(x_1 + x_2 + x_3 + x_4 - x_5 = -3m; 3) &= 93m + 94, \\
r(x_1 + x_2 + x_3 + x_4 + x_5 - x_6 = -4m; 3) &= 172m + 173.
\end{aligned}$$

The first result was also mentioned in [5] and [6]. The good colorings (that also satisfy the excellence condition) of the first two equations can be found by the accompanying program `Schaal`. The good colorings (that also satisfy the excellence condition) of the equations $x_1 + x_2 + x_3 + x_4 = x_5$ and $x_1 + x_2 + x_3 + x_4 + x_5 = x_6$ were given in [7].

3 Main Results; case $\tilde{b} > 0$

We consider the Rado numbers of (1) where the constant $\tilde{b}$ is positive. Theorem 7 gives the upper bounds and Theorem 8 gives a sufficient condition for the lower bounds.

Theorem 7. *Consider the equation $\mathcal{E}(\tilde{b}) = \mathcal{E}(b)$ of the form*

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k + b, \qquad \text{where } c_i > 0, \quad b > 0.$$

Let $s = \sum_{i=1}^{k-1} c_i - 1$. If $s|b$ and $\mathcal{E}(0)$ is t-regular then

$$r(\mathcal{E}(b); t) \leq \frac{b}{s} - \left\lceil \frac{b}{s \cdot R_C(t)} \right\rceil + 1.$$

Proof. Assume $s|b$ and $\mathcal{E}(0)$ is t-regular. Let $r = \frac{b}{s} - \left\lceil \frac{b}{s \cdot R_C(t)} \right\rceil + 1$.

We write b as $b = s\,(R_C(t) \cdot m - q)$ where $m = \left\lceil \frac{b}{s \cdot R_C(t)} \right\rceil$ and $0 \leq q \leq R_C(t) - 1$. Then $r = (R_C(t) - 1) \cdot m - q + 1$.

We show that there is no good coloring on the interval $[1, r]$.

Case 1: $m = 1$.
Then $r = b/s$. We have a trivial mono solution to $\mathcal{E}(b)$ via $x_1 = x_2 = x_3 = \cdots = x_k = r$.

Case 2: $m > 1$.
Define an injective map f from $[1, R_C(t)]$ to $[1, r]$ by

$$f(w) = (R_C(t) - w) \cdot m - q + w.$$

Notice that a tuple $(w_1, w_2, \ldots, w_{k-1}, \sum_{i=1}^{k-1} c_iw_i)$ of the equation

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k$$

is made to correspond to the tuple $(f(w_1), f(w_2), \ldots, f(w_{k-1}), f(\sum_{i=1}^{k-1} c_iw_i))$ in

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k + b.$$

Please check it yourself!!!

Now, given any coloring of α on $[1, r]$, we define the coloring χ on the interval $[1, R_C(t)]$ by

$$\chi(w) = \alpha(f(w)), \quad w = 1, 2, \ldots, R_C(t).$$

From the definition of the Rado number, any coloring on $[1, R_C(t)]$ must contain a mono tuple to $\mathcal{E}(0)$. Hence there is also a mono tuple on $[1, r]$ to $\mathcal{E}(b)$. In both cases, there is no good coloring on $[1, r]$. $\square$

The lower bounds can be stated in similar way to Theorem 4.

Theorem 8. *Consider the equation* $\mathcal{E}(\tilde{b}) = \mathcal{E}(b)$ *of the form*

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k + b, \qquad \text{where } c_i > 0, \quad b > 0. \tag{3}$$

Let $s = \sum_{i=1}^{k-1} c_i - 1$. *If* $s|b$ *and there is a coloring on the interval* $[1, n]$ *which satisfies the excellence condition then*

$$r(\mathcal{E}(b); k) \geq \frac{b}{s} - \left\lceil \frac{b}{s \cdot (n+1)} \right\rceil + 1.$$

Proof. We invoke the result of Theorem 4 by rewriting (3) in the form of (2). Since $s|b$, we can write b in the form $b = s\left[(n+1)m - q\right]$ where $m = \left\lceil \frac{b}{s \cdot (n+1)} \right\rceil$ and $0 \leq q \leq n$. Then $r = \frac{b}{s} - \left\lceil \frac{b}{s \cdot (n+1)} \right\rceil + 1 = (n+1)m - q - m + 1 = nm - q + 1$.

We show that there is a "good coloring" on the interval $[1, r-1] = [1, nm-q]$ to (3).

First we rewrite (3) as

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k + s\left[(n+1)m - q\right].$$

We then rewrite this equation again as

$$c_1\left[(n+1)m - q - x_1\right] + c_2\left[(n+1)m - q - x_2\right] + \cdots + c_{k-1}\left[(n+1)m - q - x_{k-1}\right]$$
$$= (n+1)m - q - x_k.$$

Next we add $-s(m-1)$ on both sides of the equation,

$$c_1\left[nm - q + 1 - x_1\right] + c_2\left[nm - q + 1 - x_2\right] + \cdots + c_{k-1}\left[nm - q + 1 - x_{k-1}\right]$$
$$= \left[nm - q + 1 - x_k\right] - s(m-1).$$

We let $x'_i = nm - q + 1 - x_i$ for each i. The reader sees that x'_i is x_i after reversing the interval $[1, nm-q]$. The equation after substitution is

$$c_1x'_1 + c_2x'_2 + \cdots + c_{k-1}x'_{k-1} = x'_k - s(m-1). \tag{4}$$

The next step is clear. We invoke the result from Theorem 4 (!) that there is a good coloring α on the interval $[1, mn]$ to (4). We can then make a good coloring to (3) from this interval by taking the elements 1 to $mn - q$ of α and reverse the interval. □

Below are some applications of Theorems 7 and 8.

Corollary 9. *Consider the equation $\mathcal{E}(b)$ of the form*

$$x_1 + x_2 + \cdots + x_{k-1} = x_k + b, \qquad \text{with } k \geq 2, \;\; b \geq 1 \text{ and } (k-2)|b.$$

We let $m = b/(k-2)$. Then

$$r(\mathcal{E}(b); 2) = m - \left\lceil \frac{m}{k^2 - k - 1} \right\rceil + 1.$$

Proof. The proof is the same as for Corollary 5 except that this time we apply Theorems 7 and 8. □

Note that the above result when $k = 3$ was mentioned in [1].

Corollary 10. *For $b \geq 1$,*

$$r(x + y - z = b; 3) = b - \left\lceil \frac{b}{14} \right\rceil + 1.$$

Proof. The proof is straight forward. It can be checked that the original coloring $[1, 2, 2, 1, 3, 3, 3, 3, 3, 1, 2, 2, 1]$ satisfies the excellence condition. Then we apply Theorems 7 and 8. □

This result was a part of Theorem 9.15 in [2]. Although it was wrongly claimed that $r(x + y - z = b; 3) = b - \left\lceil \frac{b-1}{14} \right\rceil$.

For the situation when the equation $\mathcal{E}(0)$ given by

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k, \qquad c_i > 0,$$

is not t-regular, the trivial bounds of the Rado numbers to

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k + b, \qquad b > 0, \; s|b,$$

are

$$\left\lceil \frac{b+1}{s+1} \right\rceil \leq r(\mathcal{E}(b); t) \leq \frac{b}{s}, \quad \text{for any } t \geq 1.$$

The mono solution for the upper bound arises from the tuple $(\frac{b}{s}, \frac{b}{s}, \ldots, \frac{b}{s})$.

4 Final Remarks

So far, our results were obtained by checking the excellence condition of each good coloring. For the 2-coloring and 3-coloring, it seems that there are always colorings of length $n = R_C(t) - 1$ to the equations

$$c_1x_1 + c_2x_2 + \cdots + c_{k-1}x_{k-1} = x_k, \qquad \text{where } c_i > 0$$

that satisfy the excellence condition. Thus it makes sense to make the following conjecture.

Conjecture 11. *For $t = 2$ or 3, fix constants $c_1, c_2, \ldots, c_{k-1}$. Consider the equation $\mathcal{E}(\tilde{b})$ of the form*

$$c_1 x_1 + c_2 x_2 + \cdots + c_{k-1} x_{k-1} = x_k + \tilde{b}, \qquad \text{where } c_i > 0.$$

Let $s = \sum_{i=1}^{k-1} c_i - 1$. If $s|\tilde{b}$ and $\mathcal{E}(0)$ is t-regular then

$$r(\mathcal{E}(\tilde{b}); t) = \begin{cases} \frac{\tilde{b}}{s} - \left\lceil \frac{\tilde{b}}{s \cdot R_C(t)} \right\rceil + 1, & \text{for } \tilde{b} > 0, \\ -\frac{\tilde{b}}{s} \cdot (R_C(t) - 1) + R_C(t), & \text{for } \tilde{b} < 0. \end{cases}$$

For t-colorings where $t \geq 4$, our Maple program is too slow to give any tangible observations. A faster program could used to verify whether this conjecture still holds.

Lastly, the reader might wonder about the other type of equations that we did not consider, i.e.

$$\sum_{i=1}^{k-1} c_i x_i = c_k x_k + b, \quad \text{where } c_i \geq 1, \quad \text{for } 1 \leq i \leq k-1 \text{ and } c_k \geq 2.$$

It turns out that the Rado numbers of these equations exhibit more complicated patterns from those discovered in this paper.

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