On the Ramsey Multiplicity of Complete Graphs

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Abstract

We define a new notation $RR_n(k_1, k_2, ..., k_r)$, the minimum number of monochromatic K_{k_i} of color $i, 1 \leq i \leq r$, in an r-edge-coloring graph K_n. We show the exact answer for $RR_n(2, k_1, k_2, ..., k_r), k_i \geq 2$ and give bounds for $RR_n(3, 3, ..., 3)$.

1 Introduction

For a party of six people, there is guaranteed a group of three acquaintances or three strangers. In terms of Graph theory, for any 2-edge-coloring of a complete graph with 6 vertices, there must be a triangle with the same color of edges.

Definition: The Ramsey number $R(k_1, k_2, ..., k_r)$ is the smallest number n such that any r-edge-coloring of a complete graph with n vertices, K_n, must contain at least one subgraph K_{k_i}, of color i, for some i, $1 \leq i \leq r$.

It is a computationally very difficult to compute the exact value of $R(m, n)$. We know only some small values of R such as $R(3, 3) = 6$ and $R(4, 4) = 18$. A famous mathematician, Paul Erdös, once jokingly made fun of the search for the values of $R(5, 5)$ and $R(6, 6)$. Also one of the most famous problems in combinatorics is to find $\lim_{n\to\infty} R(n, n)^{\frac{1}{n}}$ (if it exists).

We know six people guarantee a group of three acquaintances or three strangers. But how many groups of three acquaintances and three strangers must be in a group of n people. In this paper, we investigate this type of problem which is considered to be a quantitative version of the Ramsey number.

Definition: $RR_n(k_1, k_2, ..., k_r)$ is the minimum total numbers of the monochromatic complete graph K_{k_i} of color i, $1 \leq i \leq r$, in the r-edge-coloring of K_n.

In [6], Goodman showed a solution to $RR_n(3,3)$, the minimum number of monochromatic triangles of any 2-edge-coloring of K_n. The exact answer is

$$RR_n(3,3) = \binom{n}{3} - \lfloor \frac{n}{2} \lfloor \frac{n-1}{2} \rfloor \lceil \frac{n-1}{2} \rceil \rfloor = \frac{1}{4}\binom{n}{3} + O(n^2).$$

The solution provided below used the method called "*a system of weights*" was first introduced by Suavé in [10]. By realizing that the adjacent red and blue edges gives half of non-monochromatic triangle, we find the lower bound of $RR_n(3,3)$ by minimizing the function

$$f_n(r_1, r_2, ..., r_n) = \binom{n}{3} - \lfloor \frac{1}{2} \sum_{i=1}^{n} r_i(n-1-r_i) \rfloor,$$

where r_i is the number of red edges at vertex i.

We noticed that, to obtain the minimum, $r_i = \lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$ for each i. Hence when n is even, we can construct such a graph by first partitioning the vertices into two sets of size $\frac{n}{2}$. And then letting an edge between the two partitions be red and an edge within each partition be blue.

Later in [4] and [5], Giraud used the modified version of this method for the lower bound of $RR_n(4,4)$ and $RR_n(3,3,3)$. He showed

$RR_n(3,3,3) \geq \frac{1}{29}\binom{n}{3}$ for n large enough and

$RR_n(4,4) \geq \frac{1}{46}\binom{n}{4}$ for n large enough.

The methods used were highly technical and ad hoc. While yielding a very good result, it is unlikely that it could be extended for the general cases.

In section 2, we show the values of $RR_n(2,k), k \geq 2$ and the generalized result namely the values of $RR_n(2,k_1,k_2,...,k_r),\ k_i \geq 2$ for all i. We give bounds for $RR_n(3,3,...,3)$ in section 3.

2 $RR_n(2,k)$ and $RR_n(2,k_1,k_2,...,k_r)$

We state Turán's theorem before showing the main result of this section. The reader can find proof(s) of Turán's theorem in [1] and [3].

2.1 Turán's Theorem

Definition: The Turán graph, $T_{k-1}(n)$ is the complete $(k-1)$-partite graph on n $(\geq k-1)$ vertices whose partition sets (pairwise) differ in size at most 1.
Definition: Let $t_{k-1}(n)$ be the number of edges of $T_{k-1}(n)$.

Theorem 2.1 *(Turán 1941) An n vertex graph with the maximum number of edges that does not contain K_k as a subgraph is $T_{k-1}(n)$.*

The theorem below gives the ratio between $t_{k-1}(n)$ and the total number of edges of K_n.

Theorem 2.2 $t_{k-1}(n) = \dfrac{k-2}{k-1}\dbinom{n}{2} + O(n)$.

2.2 $RR_n(2,k)$

Theorem 2.3 $RR_n(2,k) = \dfrac{1}{2(k-1)}n^2 + O(n)$.

Proof: We want to minimize the number of red edges and blue monochromatic K_k in an 2-edge-coloring of a complete graph K_n. However, if we replace an edge of a blue monochromatic K_k with a red edge, the result could only be improved. Hence the problem is equivalent to finding the minimum number of red edges on K_n so that it does not contain any blue monochromatic K_k. Or other words, we want to find the maximum number of blue edges so that it does not contain any blue monochromatic K_k. But this is exactly Turán's theorem! Therefore,

$$\begin{aligned} RR_n(2,k) &= \binom{n}{2} - t_{k-1}(n) \\ &= (1 - \frac{k-2}{k-1})\binom{n}{2} + O(n) \\ &= \frac{n^2}{2(k-1)} + O(n). \ \Box \end{aligned}$$

Example: For $RR_n(2,3)$, the Turán graph $T_2(n)$ is a complete bipartite graph with $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ vertices in each partition. The number of blue edges in this case is $\lfloor \frac{n^2}{4} \rfloor$. The number of red edges turns out to be

$$RR_n(2,3) = \binom{n}{2} - \lfloor \frac{n^2}{4} \rfloor = \lfloor \frac{(n-1)^2}{4} \rfloor = \frac{n^2}{4} + O(n).$$

2.3 $RR_n(2, k_1, k_2, ..., k_r)$

Theorem 2.4 *(A Colored version of Turán's Theorem) An n vertex graph with the maximum number of r-colored-edges that does not contain K_{k_i} of color i as a subgraph is $T_{R(k_1,k_2,...,k_r)-1}(n)$.*

Proof: Define $T := T_{R(k_1,k_2,...,k_r)-1}(n)$. We first show a coloring of T that satisfies the property. Then we show $t_{R(k_1,k_2,...,k_r)-1}(n)$ edges is already the maximum.

Let G be an r-edge-colored complete graph with $R(k_1, k_2, ..., k_r) - 1$ vertices that does not contain any monochromatic K_{k_i} of color $i, 1 \leq i \leq r$. The graph G exists by the definition of Ramsey number. We now color the edges between partition i and j of T according to the color of e_{ij} in G. It is clear that T does not contain any monochromatic K_{k_i} of color i.

We now show that there is no graph with more edges than T which satisfies the property. Assume, for a contradiction, that H is a graph with more edges than T. By Turán's theorem, H contains $K_{R(k_1,k_2,...,k_r)}$ as a subgraph. But by definition of the Ramsey number, there is at least one monochromatic K_{k_i} of color i as a subgraph of H. $\Box$

Theorem 2.5 $RR_n(2, k_1, k_2, ..., k_r) = \frac{1}{2(R(k_1, k_2, ..., k_r) - 1)} n^2 + O(n), k_i \geq 2.$

Proof: We proceed with the same argument as theorem 2.3 and calculate the number of "no edges" in $T_{R(k_1,k_2,...,k_r)-1}(n)$.

$$\begin{aligned} RR_n(2,k_1,k_2,...,k_r) &= \binom{n}{2} - t_{R(k_1,k_2,...,k_r)-1}(n) \\ &= (1 - \frac{R(k_1,k_2,...,k_r)-2}{R(k_1,k_2,...,k_r)-1})\binom{n}{2} + O(n) \\ &= \frac{n^2}{2(R(k_1,k_2,...,k_r)-1)} + O(n). \ \square \end{aligned}$$

3 Bounds for $RR_n(3,3,...,3)$

The author fails to find the exact answer. In this section, we show upper bounds and lower bounds for $RR_n(3;r),\ r \geq 3$ (defined below).

Definition: $RR_n(k;r) := RR_n(\underbrace{k,k,...,k}_{r\text{ times}})$.

Definition: $R(k;r) := R(\underbrace{k,k,...,k}_{r\text{ times}})$.

3.1 Upper Bounds for $RR_n(3,3,...,3)$

We provide examples of a "good" edge-coloring of K_n for an upper bound of $RR_n(3;r)$.

Theorem 3.1 $RR_n(3;\ r) \leq \dfrac{1}{6(R(3;r-1)-1)^2}n^3 + O(n^2),\ r \geq 2.$

Proof: Let G be a $(r-1)$-edge-colored complete graph with $R(3;r-1)-1$ vertices that does not contain any monochromatic triangles. G is well defined by the definition of R. We now divide the graph K_n into $R(3;r-1)-1$ partitions where the size of each partition is differed at most one. We color the edges between partition i and j of K_n according to the color of e_{ij} of G and color the edges in each partition with the one color left. As a result,

$$
\begin{aligned}
RR_n(3;r) &\le \text{sum of the number of triangles in each partition of } K_n \\
&\le (R(3;r-1)-1)\binom{\lceil \frac{n}{R(3;r-1)-1} \rceil}{3} \\
&= \frac{n^3}{6(R(3;r-1)-1)^2} + O(n^2). \ \square
\end{aligned}
$$

We state the following bounds as a corollary.

$RR_n(3;2) \le \frac{n^3}{24} + O(n^2)$, since $R(3) = 3$.
$RR_n(3;3) \le \frac{n^3}{150} + O(n^2)$, since $R(3,3) = 6$.
$RR_n(3;4) \le \frac{n^3}{1536} + O(n^2)$, since $R(3,3,3) = 17$.
$RR_n(3;5) \le \frac{n^3}{15000} + O(n^2)$, since $R(3,3,3,3) \ge 51$.
$RR_n(3;6) \le \frac{n^3}{155526} + O(n^2)$, since $R(3,3,3,3,3) \ge 162$.

Note that this bound for $RR_n(3;2)$ is the actual value mentioned in the first section. Also the bound for $RR_n(3;3)$ was mentioned before by Giraud. The following famous result of the lower bound for $R(3;r)$ could be used to help in the calculation of the upper bounds of $RR_n(3;r)$.

Theorem 3.2 *(Chung 1973)* $R(3;r) \ge 3R(3;r-1) + R(3;r-3) - 3, \ r \ge 4$.

3.2 Lower Bounds for $RR_n(3,3,...,3)$

Theorem 3.3 *Assume* $r \ge 3$ *and* $RR_n(3;r-1) \ge A_{r-1}n^3 - B_{r-1}n^2$, $n \ge 1$ *then*

$RR_n(3;r) \ge A_r n^3 - B_r n^2$,
where $A_r := \frac{1}{(1+\frac{3\sqrt{2}}{r})}\frac{A_{r-1}}{r^2}$ *and* $B_r :=$ *max* $\{A_r, B_{r-1}\}$.

Proof: Assume $RR_n(3;r-1) \ge A_{r-1}n^3 + B_{r-1}n^2$.
We prove the theorem by using induction on the number of vertices, n.

Base case: $n := 1$.
$A_r n^3 - B_r n^2 \le A_r n^3 - A_r n^2 = 0$. We hence verify the base case.

Induction step:
Consider a vertex $v \in K_n$.

Let C_i be the set of vertices joined to v by an i^{th}-colored-edge, $1 \leq i \leq r$.
Let $V_i := |C_i|$, $1 \leq i \leq r$.
Let x_i be the number of i^{th}-colored-edges amongst the vertices in the set C_i.

We break our induction down into 2 cases.

Case 1: $x_1 + x_2 + \cdots + x_r > 3A_r n^2$.

$$\begin{aligned} RR_n(3;r) &> 3A_r n^2 + RR_{n-1}(3;r) \\ &\geq 3A_r n^2 + A_r(n-1)^3 - B_r(n-1)^2 \quad \text{by induction assumption} \\ &= A_r n^3 + 3A_r n - A_r - B_r(n-1)^2 \\ &> A_r n^3 - B_r n^2. \end{aligned}$$

Case 2: $x_1 + x_2 + \cdots + x_r \leq 3A_r n^2$.

We first assume, each set C_i has only $(r-1)$-colored-edges (except i^{th} color). We obtain the minimum monochromatic triangles. We later subtract by $x_i V_i$, the number of nonmonochromatic triangles x_i can create in C_i. This leads to an inequality:

$$\begin{aligned} RR_n(3;\ r) \quad &\geq \quad (A_{r-1}V_1^3 - B_{r-1}V_1^2 - x_1V_1) + (A_{r-1}V_2^3 - B_{r-1}V_2^2 - x_2V_2) \\ &\quad\quad + \ldots + (A_{r-1}V_r^3 - B_{r-1}V_r^2 - x_rV_r). \\ &= \quad A_{r-1}(V_1^3 + V_2^3 + \ldots + V_r^3) - B_{r-1}(V_1^2 + V_2^2 + \ldots + V_r^2) \\ &\quad\quad -(x_1V_1 + x_2V_2 + \ldots + x_rV_r). \end{aligned}$$

To complete the induction, we need to show that

$$A_{r-1}(V_1^3+V_2^3+\ldots+V_r^3)-B_{r-1}(V_1^2+V_2^2+\ldots+V_r^2)-(x_1V_1+x_2V_2+\ldots+x_rV_r) \geq A_r n^3 - B_r n^2.$$

However, by the definition of B_r, we know $B_r n^2 \geq B_{r-1}(V_1^2+V_2^2+\ldots+V_r^2)$, it is enough to show that

$$\begin{aligned} &f(V_1, V_2, .., V_r, x_1, x_2, ..., x_r) \\ := &A_{r-1}(V_1^3 + V_2^3 + \ldots + V_r^3) - (x_1V_1 + x_2V_2 + \ldots + x_rV_r) - A_r n^3 \geq 0, \end{aligned}$$

under conditions: $V_1 + V_2 + \ldots + V_r = n-1$ and $x_1 + x_2 + \ldots + x_r \leq 3A_r n^2$.

Let $V_1 := tn$ where $0 \leq t \leq 1$.

The minimum of f occurs when $V_2 = V_3 = ... = V_r$ and $x_1 = 3A_r n^2$. After substituting these conditions in f, we now only need to show:

$\{[t^3 + (\frac{1-t}{r-1})^3(r-1)]A_{r-1} - 3A_r t - A_r\}n^3 \geq 0.$

Or showing that $g(t) := t^3 + (\dfrac{1-t}{r-1})^3(r-1) - \dfrac{3t+1}{r(r+3\sqrt{2})} \geq 0.$

We find t^* such that $g'(t^*) = 0$ (using Maple program, t^* is in term of r). Then we show that $g(t^*) \geq 0$ (by finding the root of r after plugging in t^* in g). Since both $g(0)$ and $g(1)$ are also positive, we complete the induction of case 2 and finish the proof of the theorem. □

Corollary 3.4 *$RR_n(3;3) \geq \dfrac{1}{522}n^3 - \dfrac{1}{4}n^2$ for $n \geq 1$.*

Proof: From the result in section 1, $RR_n(3;2) \geq \frac{n^3}{24} - \frac{n^2}{4}$. The statement follows directly from theorem 3.3.

The final result shows the connection between minimum number of Schur triples and the problem in this section. More information about Schur Triples can be found in [11] and [9].

Corollary 3.5 *The minimum number, over all r-colorings of $[1,n]$, of monochromatic Schur Triples, $x + y = z$, is of order n^2.*

Proof: An upper bound follows from the fact that the number of triples of $[1,n]$ is $\lfloor \frac{n^2}{4} \rfloor$ which is of order n^2. On the other hand, the number of monochromatic triples could not have an order smaller than n^2. Otherwise we can create a graph G with the order of monochromatic triangles smaller than n^3 using the color translation method. This G gives a contradiction to theorem 3.3. Note that we can define the color translation of the triple $(i,j,i+j)$ from $[1,n]$ to the triangle $v_I v_J v_K$ of the graph G by a relation: $(e_{IJ} \iff i, e_{JK} \iff j$ and $e_{IK} \iff i+j)$ or $(e_{IJ} \iff j, e_{JK} \iff i$ and $e_{IK} \iff i+j)$. We also see that each Schur triple corresponds to at most $2n$ triangles. □

4 Final Remark

It is possible to extend the method in section 3 for the answer of $RR_n(3,4)$. However the result might be "weak" compared to the more sophisticated "a system of weights".

References

[1] Martin Aigner and Gunter M. Ziegler, *Proof from THE BOOK*, Springer, 2000.

[2] David Conlon, *On the Ramsey Multiplicity of Complete Graphs*, Combinatorica, preprinted.

[3] Reinhard Diestel, *Graph Theory*, Springer, 2000.

[4] Guy Giraud, *Sur les nombres de triangles uni et bicolores dans un tricoloriage des arêtes du n-emble*, C.R. Acad.Sci.ParisSér.A **278**, 1974, 1067-1069.

[5] Guy Giraud, *Sur le probléme de Goodman pour les quadrangles et la majoration des nombres de Ramsey*, J.Combin.Theory Ser.B **27**, 1979, no.3, 237-253.

[6] A.W.Goodman, *On sets of acquaintances and strangers at a party*, American Math. Monthly 66, 1959, 778-783.

[7] A.W.Goodman, *Triangles in a Complete Chromatic Graph with Three Colors*, Discrete Mathematics 57, 1985, 225-235.

[8] R.L.Graham, B.Rothschild and J.H.Spencer, *Ramsey Theory*, Wiley, 1990.

[9] Bruce M. Landman and Aaron Robertson, *Ramsey Theory on the Integers*, AMS, 2004.

[10] Leopold Sauv́e, *On Chromatic graphs*, Amer. Math. Monthly 68, 1961, 107-111.

[11] Thotsaporn "Aek" Thanatipanonda, *On the Monochromatic Schur Triples type problem* , Electronic Journal of Combinatorics R14 of Volume 16(1), 2009.