Rook Endgame Problems in m by n Chess

Thotsaporn "Aek" Thanatipanonda
thotsaporn@gmail.com

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Abstract

We consider Chess played on an $m \times n$ board (with m and n arbitrary positive integers), with only the two kings and the white rook remaining, but placed at arbitrary positions. Using the symbolic finite state method, developed by Thanatipanonda and Zeilberger, we prove that on a $3 \times n$ board, for almost all initial positions, White can checkmate Black in $\leq n+2$ moves, and that this upper bound is sharp. We also conjecture that for an arbitrary $m \times n$ board, with $m, n \geq 4$ (except for $(m, n) = (4, 4)$ when it equals 7), the number of needed moves is $\leq m+n$, and that this bound is also sharp.

1 Background and Introduction

Chess is arguably the most popular board game on this planet. There are numerous combinatorial problems inspired by Chess, such as the non-attacking queens problem [4] and the rich theory of rook polynomials [5] . More related to the actual game of Chess, Noam Elkies analyzed Chess endgame positions using the theory of combinatorial games [1].

Our investigation in this article is in an entirely different direction. We search for an answer to the following simply stated question:

"Find the minimal number of moves needed for White to checkmate against a perfect opponent when playing on an $m \times n$ board where the only pieces left are the two kings and the white rook" .

Of course, White always moves first.

While the question of the *minimal* number of moves needed is not that important in the usual 8×8 chess, since the FIDE rules generously allow 50 moves, so White has to be an extremely weak player not be able to win. On the other hand in Thai Chess, he (or she) is only allowed 16 moves, so one has to be much more clever, and the present article may be useful even for the very special 8×8 case.

2 Methodology

In any rook endgame position on an $m \times n$ board, White can always checkmate within $O(m+n)$ moves. The following elegant proof was kindly communicated to us by Noam

Elkies.

Proposition 2.1. *In a rook endgame on an $m \times n$ board, the winning side can checkmate within $O(m+n)$ moves.*

Proof. (Noam Elkies) Assuming both m and n exceed 2. [If $m = 2$ or $n = 2$ there are no checkmates at all; if $m = 1$ or $n = 1$ then either the Black King is already checkmated or no checkmate is possible – unless White castles on move 1...] It is true that the method taught in most chess manuals takes time proportional to $m \cdot n$ moves. But this can be reduced to $O(m+n)$ by using the Rook, supported by the King, to restrict the opposing King to an $a \times b$ rectangle ($a < m, b < n$): it takes only $O(1)$ moves to tighten the noose in one dimension or the other, decrementing a to $a-1$ or b to $b-1$; in $O(m+n)$ moves, then, the King will be limited to $O(1)$ squares near the corner, and then checkmate follows in another $O(1)$ moves. □

It would be desirable to have explicit, and if possible, sharp, upper bounds for the needed number of moves. We use a symbolic finite state method analogous to the one used in [3].

Let us emphasize the importance of the *methodology*, that far transcends the actual results. i.e. that of rigorous experimental mathematics. We have a fully implemented algorithm, that completely does the proof without any human intervention!

The computer first generates the data from many $k \times n$ boards, for *numeric* k and n. Then it makes *symbolic* conjectures. Finally the computer automatically proves these general (symbolic) conjectures *all by itself*. We believe that this style of proof will be more commonly used in the future.

3 Detailed Example

In this section, we give an example of how the problem could be done on a $3 \times n$ board. However, it becomes much harder to handle wider widths.

3.1 Example on a $3 \times n$ board

We fix some "natural" positions on a $3 \times n$ board given by figure 1.

The black King must be above the white King and the white King must be above the Rook, as shown in the diagram.
a is the distance between the black King and the white King.
b is the distance of the black King to the end of the board.
c is the distance of the Rook to the other end of the board, but is irrelevant in the calculation.

Let x be the column of the black King.
Let y be the column of the white King.
Let z be the column of the white Rook.
Let $f_{x,y,z}(a, b)$ be the number of moves needed to checkmate with the above initial position.

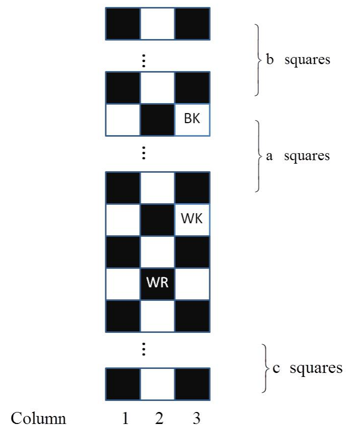

Figure 1: Standard Position on a $3 \times n$ Board

We gives sharp upper bounds for all possible $f_{x,y,z}(a,b), a \geq 0, b \geq 0$. The proof comprises of three steps, generating data, making conjectures and proving conjectures.

The claims are as follows.

$$
\begin{aligned}
f_{1,1,1}(a,b) &\leq a+b+1 \quad , a \geq 2. \\
f_{1,1,2}(a,b) &\leq \begin{cases} a+b+1 & ,a \text{ is even and } a \geq 2. \\ a+b & ,a \text{ is odd and } a \geq 2. \end{cases} \\
f_{1,1,3}(a,b) &\leq a+b+1 \quad , a \geq 2. \\
f_{1,2,2}(a,b) &\leq a+b \quad , a \geq 2. \\
f_{1,2,3}(a,b) &\leq \begin{cases} a+b+2 & ,a \text{ is even and } a \geq 2. \\ a+b+1 & ,a \text{ is odd and } a \geq 2. \end{cases} \\
f_{1,3,2}(a,b) &\leq \begin{cases} 1 & ,a = 0. \\ a+b & ,a \text{ is even and } a \geq 1. \\ a+b+1 & ,a \text{ is odd and } a \geq 1. \end{cases} \\
f_{1,3,3}(a,b) &\leq \begin{cases} 1 & ,a = 0. \\ a+b+2 & ,a \text{ is even and } a \geq 1. \\ a+b+1 & ,a \text{ is odd and } a \geq 1. \end{cases} \\
f_{2,1,1}(a,b) &\leq a+b+2 \quad , a \geq 2. \\
f_{2,1,3}(a,b) &\leq \begin{cases} a+b+2 & ,a \text{ is even and } a \geq 2. \\ a+b+1 & ,a \text{ is odd and } a \geq 2. \end{cases} \\
f_{2,2,1}(a,b) &\leq a+b+1 \quad , a \geq 2. \\
f_{2,2,2}(a,b) &\leq \begin{cases} a+b+1 & ,a \text{ is even and } a \geq 2. \\ a+b & ,a \text{ is odd and } a \geq 2. \end{cases}
\end{aligned}
$$

Once we got the right set-up, the proof by induction on $a+b$ falls through naturally.

In order to illustrate our method, we will now present all the details, in a humanly-readable prose, for the inductive step for the first assertion above i.e. $f_{1,1,1}(a,b) \leq a+b+1$ $(a \geq 2)$. Of course, this proof was originally discovered by the computer, and the computer quickly does *all* the cases. Once discovered, the computer proof is almost instantaneous, but *discovering* the proposed proof takes longer.

Proof. **Base Case:** Verify that all the conjectures are true for $a+b \leq 3$ where $a \geq 0, b \geq 0$.

Induction Step: Consider $f_{1,1,1}(a, b)$;

Case 1: a is even.

White chooses to move his Rook to the second column. Black's only legal moves are to move his King up or down the first column. By the inductive hypothesis, in case the black King moves up, it would take at most $f_{1,1,2}(a+1, b-1) = (a+1) + (b-1) = a+b$ moves to checkmate. In case the black King moves down, it would take at most $f_{1,1,2}(a-1, b+1) = (a-1) + (b+1) = a+b$ moves to checkmate. Therefore case 1 takes at most $(a+b)+1$ moves to checkmate.

Case 2: a is odd.

White chooses to move his King in an up-right direction. Then the black King will be in check and must move to the second column. By the induction assumption, it would take at most $\max\{f_{2,2,1}(a-2, b+1), f_{2,2,1}(a-1, b), f_{2,2,1}(a, b-1)\} = (a+b-1)+1$ moves to checkmate. Therefore this case takes $(a-1+b)+1+1 = a+b+1$ moves to checkmate.

The proof of case $f_{1,1,1}(a, b)$ is done. The rest of the proof is left to the readers, if they wish, but they may prefer to look at the (humanly readable!) computer's full proof at
`http://thotsaporn.com/Rook.html` .

□

3.2 The Maple programs

The present method was implemented using Maple. The two programs, as well as sample output files, are freely available from:
`http://thotsaporn.com/Rook.html`

4 Conjecture

We end this article with an ultimate conjecture suggested strongly by numerical evidences for small m and n, given in table 1.

Let $U(m, n)$ be the maximum of the minimal number of moves needed to checkmate from any initial position of a rook endgame on an $m \times n$ board.

Conjecture 4.1. *Let m, n be integer such that $m \geq 4$ and $n \geq 4$ except $(m, n) = (4, 4)$. In a rook endgame on an $m \times n$ board, the winning side can checkmate within $m+n$ moves, and this upper bound is sharp.*

$m \backslash n$	3	4	5	6	7	8	9	10	11	12	13
3	3	5	7	8	9	10	11	12	13	14	15
4	-	7	9	10	11	12	13	14	15	16	17
5	-	-	10	11	12	13	14	15	16	17	18
6	-	-	-	12	13	14					
7	-	-	-	-	14	15					
8	-	-	-	-	-	16					

Table 1: Values of $U(m, n)$

5 Acknowledgment

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References

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[4] Wikipedia, *Eight Queens Puzzle*, http://en.wikipedia.org/wiki/Eight_queens_puzzle.

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