Schimdt type problem solved by AI

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Abstract. Around 2012, I gave a simple solution to Schmidt's Conjecture (1992) which state that: For any integer $r \geq 1$, the sequence of numbers $\{c_j^{(r)}\}_{j\geq 0}$ is defined implicitly by

$$\sum_k \binom{n}{k}^r \binom{n+k}{k}^r = \sum_j \binom{n}{j}\binom{n+j}{j} c_j^{(r)}, \quad n = 0, 1, 2, \ldots$$

Asmus Schmidt conjectured that all $c_j^{(r)}$ are integers.

Afterwards, I explored this type of problem and made my own conjecture:
Conjecture: For any integers d, $k \geq 0$ and $r \geq 1$, there exist integers $a_{d,k,j}^{(r)}$ such that

$$\binom{n+dk}{k}^r = \sum_j a_{d,k,j}^{(r)} \binom{n+dj}{j} \quad \text{for all } n = 0, 1, 2, \ldots.$$

Nobody cares about this for a long time. Only recently, two Ph.D. students from NUS (National University of Singapore), named Wenlin Zhang and Haobo, showed me the beautiful solution by AI. We will discuss about it in this talk.

I. Schimdt's Conjecture

For any integer $r \geq 1$, the sequence of numbers $\{c_j^{(r)}\}_{j\geq 0}$ is defined implicitly by

$$\sum_k \binom{n}{k}^r \binom{n+k}{k}^r = \sum_j \binom{n}{j}\binom{n+j}{j} c_j^{(r)}, \quad n = 0, 1, 2, \ldots$$

In 1992, Asmus Schmidt [4] conjectured that all $c_j^{(r)}$ are integers.

Background:

In Concrete Mathematics [1] on page 256, it was stated as a research problem. Already here, it was indicated that H. Wilf had shown the integrality of $c_n^{(r)}$ for any r but only for $n \leq 9$. For the first nontrivial case, $r = 2$; $\sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$ are the famous Apéry numbers, the denominators of rational approximations to $\zeta(3)$. This case was proved in 1992 independently by Schmidt himself [5] and by Strehl [6]. They both gave an explicit expression for $c_n^{(2)}$

$$c_n^{(2)} = \sum_j \binom{n}{j}^3 = \sum_j \binom{n}{j}^2 \binom{2j}{n}.$$

These numbers are called Franel numbers.

In the same paper [6], Strehl also gave a proof for $r = 3$ which uses Zeilberger's algorithm of creative telescoping. He also gave an explicit expression for $c_n^{(3)}$

$$c_n^{(3)} = \sum_j \binom{n}{j}^2 \binom{2j}{j}^2 \binom{2j}{n-j}.$$

The first full proof was given by Zudilin [9] in 2004 using a multiple generalization of Whipple's transformation for hypergeometric functions. Since then, the congruence properties related to the Schmidt numbers $S_n^{(r)} := \sum_k \binom{n}{k}^r \binom{n+k}{k}^r$ and to the Schmidt polynomials $S_n^{(r)}(x) := \sum_k \binom{n}{k}^r \binom{n+k}{k}^r x^k$ have been studied extensively.

Here, we present a simple proof to Schmidt's original problem in [7].

It is a natural first step to investigate the individual term $\binom{n}{k}^r \binom{n+k}{k}^r$ before considering the full sum $\sum_k \binom{n}{k}^r \binom{n+k}{k}^r$. Our proof rests on the following lemma, which was first proved by Guo and Zeng [3, 2]. Although, in this talk, we give a simple, well motivated, computer proof of their lemma.

Lemma 1. *(Guo and Zeng, 2012) For $k \geq 0$ and $r \geq 1$, there exist integers $a_{k,j}^{(r)}$ with $a_{k,j}^{(r)} = 0$ for $j < k$ or $j > rk$, and*

$$\binom{n}{k}^r \binom{n+k}{k}^r = \sum_j a_{k,j}^{(r)} \binom{n}{j} \binom{n+j}{j}, \quad n = 0, 1, 2, 3, \ldots. \tag{1}$$

Remark: Schimdt's conjecture can easily be seen true from this Lemma.

$$\sum_k \binom{n}{k}^r \binom{n+k}{k}^r = \sum_k \sum_j a_{k,j}^{(r)} \binom{n}{j} \binom{n+j}{j} = \sum_j \binom{n}{j} \binom{n+j}{j} \sum_k a_{k,j}^{(r)}.$$

Therefore, we have

$$c_j^{(r)} = \sum_k a_{k,j}^{(r)}.$$

Proof of Lemma 1:

Hidden structure!

Define $\bar{a}_{k,j}^{(r)}$ recursively by $\bar{a}_{k,k}^{(1)} = 1$, $\bar{a}_{k,j}^{(1)} = 0$ $(j \neq k)$ and

$$\bar{a}_{k,j}^{(r+1)} = \sum_i S(k,j,i) \cdot \bar{a}_{k,i}^{(r)}, \tag{2}$$

where $S(k,j,i)$ is independent of r and is an integer. Then it is clear that $\bar{a}_{k,j}^{(r)}$ are integers.

Let's say such $S(k,j,i)$ that satisfies (2) exists. Then we show by induction on r that $\bar{a}_{k,j}^{(r)}$ satisfies (1).

The statement is clearly true for $r = 1$. The induction steps,

Left hand side:

$$\binom{n}{k}^{r+1}\binom{n+k}{k}^{r+1} = \binom{n}{k}^{r}\binom{n+k}{k}^{r}\binom{n}{k}\binom{n+k}{k}$$

(by induction hypothesis)

$$= \sum_i \bar{a}_{k,i}^{(r)}\binom{n}{i}\binom{n+i}{i}\binom{n}{k}\binom{n+k}{k}.$$

Right hand side:

$$\sum_j \bar{a}_{k,j}^{(r+1)}\binom{n}{j}\binom{n+j}{j} = \sum_j\sum_i \bar{a}_{k,i}^{(r)} S(k,j,i)\binom{n}{j}\binom{n+j}{j}$$

(by definition of $\bar{a}_{k,j}^{(r+1)}$)

$$= \sum_i \bar{a}_{k,i}^{(r)} \sum_j S(k,j,i)\binom{n}{j}\binom{n+j}{j}.$$

Hence things are boiling down to show that

$$\sum_j S(k,j,i)\binom{n}{j}\binom{n+j}{j} = \binom{n}{i}\binom{n+i}{i}\binom{n}{k}\binom{n+k}{k}.$$

We can guess $S(k,j,i)$ through fitting the holonomic relation that

$$S(k,j,i) = \binom{k+i}{i}\binom{k}{j-i}\binom{j}{k}.$$

That is we can prove Lemma 1 by showing that

$$\sum_j \binom{k+i}{i}\binom{k}{j-i}\binom{j}{k}\binom{n}{j}\binom{n+j}{j} = \binom{n}{i}\binom{n+i}{i}\binom{n}{k}\binom{n+k}{k}, \qquad (3)$$

which, luckily, can be verified with Zeilberger's algorithm.

Question 1: Find the combinatorial proof of (3).

II. Generalization

After I was done with Schimdt's conjecture/problem, I started to search for a function of similar type to work with, i.e. a binomial that is in the form

$$B_k(n)^r = \sum_j a_{k,j}^{(r)} \cdot B_j(n), \tag{4}$$

where $a_{k,j}^{(r)}$ are integers.

I was hoping to turn the observations to rigorous statement. The first step is to *test* for relation

$$a_{k,j}^{(r+1)} = \sum_i S(k,j,i) \cdot a_{k,i}^{(r)},$$

for some integer function $S(k,j,i)$ that independent of r.

Similar to Schimdt's conjecture/problem, the proof of (4) boils down to show that

$$B_i(n) \cdot B_k(n) = \sum_j S(k,j,i) B_j(n), \tag{5}$$

for all $n, i, k \geq 0$.

Discussion:

In the process of discovery, we went from (4) and things were boiling down to (5).

On the other hand, to prove Theorem/Lemma formally, we started by defining $S(k, j, i)$, and used it to verify (5). Then we work all the way up. The value of $a_{k,j}^{(r)}$ is defined recursively from $S(k, j, i)$. Then this will automatically prove (4).

Conjecture 1. For any integers d, $k \geq 0$ and $r \geq 1$, there exist integers $a_{d,k,j}^{(r)}$ such that

$$\binom{n+dk}{k}^r = \sum_j a_{d,k,j}^{(r)} \binom{n+dj}{j} \quad \text{for all } n = 0, 1, 2, ...$$

Moreover $a_{d,k,j}^{(r)}$ can be defined as following: $a_{d,k,k}^{(1)} = 1$, $a_{d,k,j}^{(1)} = 0$ for $j \neq k$ and

$$a_{d,k,j}^{(r+1)} = \sum_i S_d(k,j,i) a_{d,k,i}^{(r)},$$

where, $S_d(k,j,i)$ are integers, independent of r, for all d, k, j, i.

Remark: I got this conjecture proved for case $d = 0$ and $d = 1$ only. With $S(k,j,i) = \binom{j}{k}\binom{k}{j-i}$ and $S(k,j,i) = (-1)^{i+j+k}\binom{j}{k}\binom{k}{j-i}$, respectively.

III. AI's Solution

The rest of the conjecture has been open for many years until *Wenlin Zhang* and *Haobo* (around October 9, 2026) showed me the solution from AI. **Note:** I also put AI's solution on my website.

I will give the outline of the proof here.

Step 1 (Inverse of the basis): By Vandermonde's identity:

$$\binom{n+dt}{t} = \sum_{v=0}^{t} \binom{dt}{t-v}\binom{n}{v}.$$

The inverse basis of this identity is given as

$$\binom{n}{t} = \sum_{j=0}^{t} W_{t,j}\binom{n+dj}{j},$$

where

$$W_{t,j} = \begin{cases} 1 & ,j = t \\ -\sum_{v=0}^{t-1}\binom{dt}{t-v}\cdot W_{v,j} & ,j \neq t. \end{cases}$$

It is clear that $W_{t,j}$ is always an integer. Also, $W_{t,j} = 0$ when $j > t$ or $j < 0$.

Note: In term of matrix notation: Let $U = [\binom{dt}{t-v}]_{t,v}$ and $W = [W_{t,j}]_{t,j}$.

Then $W = U^{-1}$. Both U and W are uni-triangular matrices.

Step 2 (Important identity): For all $n, p, q \in \mathbb{N}$,

$$\binom{n}{p}\binom{n}{q} = \sum_{u=0}^{p} \binom{n}{q+u}\binom{q+u}{p}\binom{p}{u}. \tag{6}$$

Indeed, the left side counts ordered pairs (P, Q) of subsets of an n-element set with $|P| = p$ and $|Q| = q$. Classify them by $u = |P \setminus Q|$, so $|P \cup Q| = q + u$. Choose the union, then choose P inside it, then choose the u elements of P absent from Q. These choices determine Q uniquely and give the three factors on the right.

Step 3 (Defining $S_d(k,j,i)$):

Define $S_d(k,j,i)$ as

$$S_d(k,j,i) = \sum_{p=0}^{k} \sum_{q=0}^{i} \sum_{u=0}^{p} \binom{dk}{k-p} \binom{di}{i-q} \binom{q+u}{p} \binom{p}{u} W_{q+u,j}.$$

Proof. (Conjecture 1)

We only need to verify (5) which now read

$$\binom{n+di}{i} \cdot \binom{n+dk}{k} = \sum_{j=0}^{i+k} S(k,j,i) \binom{n+dj}{j}. \tag{7}$$

The proof is now straight forward. What impressed me is how *one* "guess" the correct function $S_d(k,j,i)$ in the first place.

Let me show the proof anyway. Start from the left hand side,

$$
\begin{aligned}
&\sum_{j=0}^{i+k} S(k,j,i)\binom{n+dj}{j} \\
&= \sum_{p=0}^{k}\sum_{q=0}^{i}\sum_{u=0}^{p}\binom{dk}{k-p}\binom{di}{i-q}\binom{q+u}{p}\binom{p}{u}\sum_{j=0}^{q+u} W_{q+u,j}\binom{n+dj}{j}, \quad \text{by Step 3} \\
&= \sum_{p=0}^{k}\sum_{q=0}^{i}\sum_{u=0}^{p}\binom{dk}{k-p}\binom{di}{i-q}\binom{n}{q+u}\binom{q+u}{p}\binom{p}{u}, \quad \text{by Step 1} \\
&= \sum_{p=0}^{k}\sum_{q=0}^{i}\binom{dk}{k-p}\binom{di}{i-q}\binom{n}{p}\binom{n}{q}, \quad \text{by Step 2} \\
&= \left[\sum_{p=0}^{k}\binom{dk}{k-p}\binom{n}{p}\right]\cdot\left[\sum_{q=0}^{i}\binom{di}{i-q}\binom{n}{q}\right] \\
&= \binom{n+dk}{k}\binom{n+di}{i}, \quad \text{by Vandermonde's identity .}
\end{aligned}
$$

□

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