A Simple Proof of Schmidt's Conjecture

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Abstract

Using Difference Equations and Zeilberger's algorithm, we give a very simple proof of a conjecture of Asmus Schmidt that was first proved by Zudilin.

For any integer $r \geq 1$, the sequence of numbers $\{c_k^{(r)}\}_{k\geq 0}$ is defined implicitly by

$$\sum_k \binom{n}{k}^r \binom{n+k}{k}^r = \sum_k \binom{n}{k}\binom{n+k}{k} c_k^{(r)}, \quad n = 0, 1, 2, \ldots$$

In 1992, Asmus Schmidt [4] conjectured that all $c_k^{(r)}$ are integers. In Concrete Mathematics [1] on page 256, it was stated as a research problem. Already here, it was indicated that H. Wilf had shown the integrality of $c_n^{(r)}$ for any r but only for $n \leq 9$. For the first nontrivial case, $r = 2$; $\sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$ are the famous Apéry numbers, the denominators of rational approximations to $\zeta(3)$. This case was proved in 1992 independently by Schmidt himself [5] and by Strehl [6]. They both gave an explicit expression for $c_n^{(2)}$

$$c_n^{(2)} = \sum_j \binom{n}{j}^3 = \sum_j \binom{n}{j}^2 \binom{2j}{n}.$$

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These numbers are called Franel numbers. In the same paper [6], Strehl also gave a proof for $r = 3$ which uses Zeilberger's algorithm of creative telescoping. He also gave an explicit expression for $c_n^{(3)}$

$$c_n^{(3)} = \sum_j \binom{n}{j}^2 \binom{2j}{j}^2 \binom{2j}{n-j}.$$

The first full proof was given by Zudilin [7] in 2004 using a multiple generalization of Whipple's transformation for hypergeometric functions. Since then, the congruence properties related to the Schmidt numbers $S_n^{(r)} := \sum_k \binom{n}{k}^r \binom{n+k}{k}^r$ and to the Schmidt polynomials $S_n^{(r)}(x) := \sum_k \binom{n}{k}^r \binom{n+k}{k}^r x^k$ have been studied extensively. In this note, we return to Schmidt's original problem and present a simple proof.

It is a natural first step to investigate the individual term $\binom{n}{k}^r \binom{n+k}{k}^r$ before considering the full sum $\sum_k \binom{n}{k}^r \binom{n+k}{k}^r$. Our proof rests on the following lemma, which was proved by Guo and Zeng [3, 2]. In order to keep this note self-contained, we give a simple, well motivated, computer proof of their lemma.

Lemma. *For $k \geq 0$ and $r \geq 1$, there exist integers $a_{k,j}^{(r)}$ with $a_{k,j}^{(r)} = 0$ for $j < k$ or $j > rk$, and*

$$\binom{n}{k}^r \binom{n+k}{k}^r = \sum_j a_{k,j}^{(r)} \binom{n}{j} \binom{n+j}{j} \tag{1}$$

for all $n \geq 0$.

Proof. Define $\bar{a}_{k,j}^{(r)}$ recursively by $\bar{a}_{k,k}^{(1)} = 1$, $\bar{a}_{k,j}^{(1)} = 0$ $(j \neq k)$ and

$$\bar{a}_{k,j}^{(r+1)} = \sum_i \binom{k+i}{i} \binom{k}{j-i} \binom{j}{k} \bar{a}_{k,i}^{(r)}. \tag{2}$$

Then it is clear that $\bar{a}_{k,j}^{(r)}$ are integers.

We show by induction on r that $\bar{a}_{k,j}^{(r)}$ satisfies (1). The statement is clearly true for $r = 1$. Suppose the statement is true for r. Then

$$\begin{aligned}
\sum_j \bar{a}_{k,j}^{(r+1)} \binom{n}{j}\binom{n+j}{j} &= \sum_j \sum_i \bar{a}_{k,i}^{(r)} \binom{k+i}{i}\binom{k}{j-i}\binom{j}{k}\binom{n}{j}\binom{n+j}{j} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{by definition of } \bar{a}_{k,j}^{(r+1)}) \\
&= \sum_i \bar{a}_{k,i}^{(r)} \sum_j \binom{k+i}{i}\binom{k}{j-i}\binom{j}{k}\binom{n}{j}\binom{n+j}{j} \\
&= \sum_i \bar{a}_{k,i}^{(r)} \binom{n}{i}\binom{n+i}{i}\binom{n}{k}\binom{n+k}{k} \\
&= \binom{n}{k}^r \binom{n+k}{k}^r \binom{n}{k}\binom{n+k}{k} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{by induction hypothesis}) \\
&= \binom{n}{k}^{r+1} \binom{n+k}{k}^{r+1}.
\end{aligned}$$

The identity from line 2 to line 3,

$$\binom{n}{i}\binom{n+i}{i}\binom{n}{k}\binom{n+k}{k} = \sum_j \binom{k+i}{i}\binom{k}{j-i}\binom{j}{k}\binom{n}{j}\binom{n+j}{j},$$

can be verified easily with Zeilberger's algorithm.

Therefore $\bar{a}_{k,j}^{(r)}$ satisfies (1). For the lemma, we can now take $a_{k,j}^{(r)} = \bar{a}_{k,j}^{(r)}$. $\square$

The definition (2) may seem to come out of nowhere. It was found as follows. We tried to find a relation of the form:

$$a_{k,j}^{(r+1)} = \sum_i s(k,j,i) a_{k,i}^{(r)}.$$

with the hope to find a nice formula for $s(k,j,i)$, free of r. The coefficients $s(k,j,i)$ then were found by automated guessing. First we calculated the numbers $a_{k,j}^{(r)}$ for r from 1 to 15 and all k, j. Then we made an ansatz for a hypergeometric term $s(k,j,i)$. Fitting this ansatz to the calculated data and solving the constants led to the conjecture

$$s(k,j,i) = \binom{k+i}{i}\binom{k}{j-i}\binom{j}{k}.$$

Now we give a proof of the main statement. By the lemma, we have

$$\sum_i \binom{n}{i}^r \binom{n+i}{i}^r = \sum_i \sum_k a_{i,k}^{(r)} \binom{n}{k} \binom{n+k}{k} = \sum_k \binom{n}{k} \binom{n+k}{k} \sum_i a_{i,k}^{(r)}.$$

Therefore, we have

$$c_k^{(r)} = \sum_i a_{i,k}^{(r)}.$$

which concludes our statement.

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