A Simple Proof of Schmidt’s Conjecture

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Abstract

Using Difference Equations and Zeilberger’s algorithm, we give a very simple proof of a conjecture of Asmus Schmidt that was first proved by Zudilin.

For any integer \( r \geq 1 \), the sequence of numbers \( \{c_k^{(r)}\}_{k \geq 0} \) is defined implicitly by

\[
\sum_k \binom{n}{k}^r \binom{n+k}{k}^r = \sum_k \binom{n}{k}^r \binom{n+k}{k} c_k^{(r)}, \quad n = 0, 1, 2, \ldots
\]

In 1992, Asmus Schmidt [4] conjectured that all \( c_k^{(r)} \) are integers. In Concrete Mathematics [1] on page 256, it was stated as a research problem. Already here, it was indicated that H. Wilf had shown the integrality of \( c_n^{(r)} \) for any \( r \) but only for \( n \leq 9 \). For the first nontrivial case, \( r = 2 \): \( \sum_k (\binom{n}{k})^2 (\binom{n+k}{k})^2 \) are the famous Apéry numbers, the denominators of rational approximations to \( \zeta(3) \). This case was proved in 1992 independently by Schmidt himself [5] and by Strehl [6]. They both gave an explicit expression for \( c_n^{(2)} \)

\[
c_n^{(2)} = \sum_j \binom{n}{j}^3 = \sum_j \binom{n}{j}^2 \binom{2j}{n}.
\]

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These numbers are called Franel numbers. In the same paper [6], Strehl also gave a proof for \( r = 3 \) which uses Zeilberger’s algorithm of creative telescoping. He also gave an explicit expression for \( c_n^{(3)} \):

\[
c_n^{(3)} = \sum_j \left( \frac{n}{j} \right)^2 \left( \frac{2j}{j} \right)^2 \left( \frac{2j}{n-j} \right).
\]

The first full proof was given by Zudilin [7] in 2004 using a multiple generalization of Whipple’s transformation for hypergeometric functions. Since then, the congruence properties related to the Schmidt numbers \( S_n^{(r)} := \sum_k \binom{n}{k}^r \binom{n+k}{k}^r \) and to the Schmidt polynomials \( S_n^{(r)}(x) := \sum_k \binom{n}{k}^r \binom{n+k}{k}^r x^k \) have been studied extensively. In this note, we return to Schmidt’s original problem and present a simple proof.

It is a natural first step to investigate the individual term \( \binom{n}{k}^r \binom{n+k}{k}^r \) before considering the full sum \( \sum_k \binom{n}{k}^r \binom{n+k}{k}^r \). Our proof rests on the following lemma, which was proved by Guo and Zeng [3, 2]. In order to keep this note self-contained, we give a simple, well motivated, computer proof of their lemma.

**Lemma.** For \( k \geq 0 \) and \( r \geq 1 \), there exist integers \( a_{k,j}^{(r)} \) with \( a_{k,j}^{(r)} = 0 \) for \( j < k \) or \( j > rk \), and

\[
\binom{n}{k}^r \binom{n+k}{k}^r = \sum_j a_{k,j}^{(r)} \binom{n}{j}^r \binom{n+j}{j}^r
\]

for all \( n \geq 0 \).

**Proof.** Define \( \bar{a}_{k,j}^{(r)} \) recursively by \( \bar{a}_{k,k}^{(1)} = 1 \), \( \bar{a}_{k,j}^{(1)} = 0 \) \((j \neq k)\) and

\[
\bar{a}_{k,j}^{(r+1)} = \sum_i \binom{k+i}{i} \binom{k}{j-i} \binom{j}{k} \bar{a}_{k,i}^{(r)}.
\]

Then it is clear that \( \bar{a}_{k,j}^{(r)} \) are integers.

We show by induction on \( r \) that \( \bar{a}_{k,j}^{(r)} \) satisfies (1). The statement is clearly
true for \( r = 1 \). Suppose the statement is true for \( r \). Then

\[
\sum_j \bar{a}_{k,j}^{(r+1)} \binom{n}{j} \binom{n+j}{j} = \sum_j \sum_i \bar{a}_{k,i}^{(r)} \binom{k+i}{i} \binom{k}{j-i} \binom{j}{k} \binom{n}{j} \binom{n+j}{j} \quad \text{(by definition of } \bar{a}_{k,j}^{(r+1)} \text{)}
\]

\[
= \sum_i \bar{a}_{k,i}^{(r)} \sum_j \binom{k+i}{i} \binom{k}{j-i} \binom{j}{k} \binom{n}{j} \binom{n+j}{j}
\]

\[
= \sum_i \bar{a}_{k,i}^{(r)} \binom{n+i}{i} \binom{n+i}{k} \binom{n}{k}
\]

\[
= \binom{n}{k} \binom{n+k}{k} \binom{n+k}{k} \binom{n+k}{k} \quad \text{(by induction hypothesis)}
\]

\[
= \binom{n}{k} \binom{n+k}{k} \binom{n+k}{k} \binom{n+k}{k} \binom{n+k}{k}
\]

The identity from line 2 to line 3,

\[
\binom{n}{i} \binom{n+i}{i} \binom{n+k}{k} \binom{n+k}{k} = \sum_j \binom{k+i}{j} \binom{k}{j-i} \binom{j}{k} \binom{n}{j} \binom{n+j}{j}
\]

can be verified easily with Zeilberger’s algorithm.

Therefore \( \bar{a}_{k,j}^{(r)} \) satisfies (1). For the lemma, we can now take \( a_{k,j}^{(r)} = \bar{a}_{k,j}^{(r)} \).

The definition (2) may seem to come out of nowhere. It was found as follows. We tried to find a relation of the form:

\[
a_{k,j}^{(r+1)} = \sum_i s(k,j,i) a_{k,i}^{(r)}
\]

with the hope to find a nice formula for \( s(k,j,i) \), free of \( r \). The coefficients \( s(k,j,i) \) then were found by automated guessing. First we calculated the numbers \( \bar{a}_{k,j}^{(r)} \) for \( r \) from 1 to 15 and all \( k,j \). Then we made an ansatz for a hypergeometric term \( s(k,j,i) \). Fitting this ansatz to the calculated data and solving the constants led to the conjecture

\[
s(k,j,i) = \binom{k+i}{i} \binom{k}{j-i} \binom{j}{k}
\]
Now we give a proof of the main statement. By the lemma, we have

\[ \sum_i \binom{n}{i}^r \binom{n+i}{i}^r = \sum_i \sum_k a_{i,k}^{(r)} \binom{n}{k} \binom{n+k}{k} = \sum_k \binom{n}{k} \binom{n+k}{k} \sum_i a_{i,k}^{(r)}. \]

Therefore, we have

\[ c_k^{(r)} = \sum_i a_{i,k}^{(r)}. \]

which concludes our statement.

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**References**


