1 Solution to von Neumann's model

Assume each player put 1 into the pot.

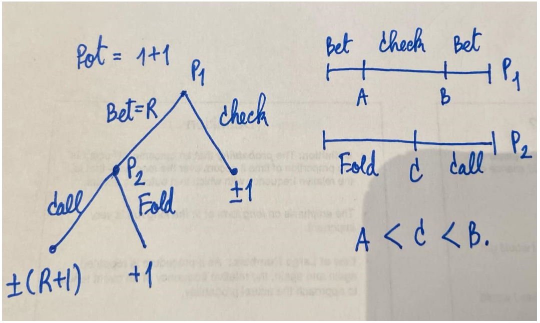

Figure 1: The betting tree and Nash equilibrium strategies for von Neumann Poker

Assume $A < C < B$. The payoff is

$$P = \int_0^A C-(R+1)(1-C)\,dx + \int_A^B x-(1-x)\,dx + \int_B^1 C+(R+1)(x-C)-(R+1)(1-x)\,dx.$$

Use calculus to solve for the optimal solution.

$$\frac{\partial P}{\partial A} = 0 \implies C-(R+1)(1-C) = 2A-1$$
$$\frac{\partial P}{\partial B} = 0 \implies 2RB = R(1+C)$$
$$\frac{\partial P}{\partial C} = 0 \implies (R+2)A+1 = B+(R+1)(1-B)$$
$$\frac{\partial P}{\partial R} = 0 \implies (C+1)(1-B)+B^2+(1-C)A = 1$$

We got $A = 1/9, B = 7/9, C = 5/9$ and $R = 2$. The payoff is $1/9$.

2 Solution to Newman's model

This time the bet $R_i \in (0, \infty)$.

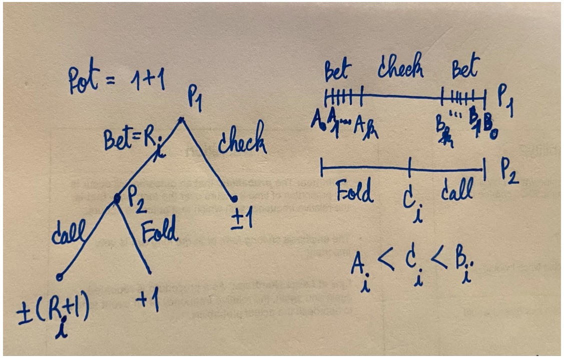

Figure 2: The betting tree and Nash equilibrium strategies for Newman Poker

For $1 \leq i \leq k$, let $R_i > 0$ be the raise on (A_{i-1}, A_i) and on (B_{i-1}, B_i) for player 1. The correspond strategy to raise R_i is C_i (as in the picture) for player 2. We also let $A_0 = 0, B_0 = 1, A_k = A, B_k = B$ and $C_k = C$.

Assume $A_i < C_i < B_i$ for each i. The payoff is

$$P = \sum_{i=1}^{k} \int_{A_{i-1}}^{A_i} C_i - (R_i + 1)(1 - C_i)\, dx + \int_{A_k}^{B_k} x - (1 - x)\, dx$$
$$+ \sum_{i=1}^{k} \int_{B_i}^{B_{i-1}} C_i + (R_i + 1)(x - C_i) - (R_i + 1)(1 - x)\, dx.$$

We use calculus to find the relation of A_i, B_i, C_i. Some of the values and relations can be found right away.

Note $R_k = 0$.

$$\frac{\partial P}{\partial A_k} = 0 \implies C_k - (R_k + 1)(1 - C_k) = 2A_k - 1 \implies A_k = C_k.$$

$$\frac{\partial P}{\partial B_k} = 0 \implies 2B_k - 1 = C_k + B_k - C_k - 1 + B_k \implies 0 = 0.$$

For each i, $1 \leq i \leq k-1$,

$$\frac{\partial P}{\partial A_i} = 0 \implies C_i - (R_i + 1)(1 - C_i) = C_{i+1} - (R_{i+1} + 1)(1 - C_{i+1}).$$

Solving this successively to get

$$C_i - (R_i + 1)(1 - C_i) = C_k - (R_k + 1)(1 - C_k) \implies C_i = \frac{R_i + 2C_k}{R_i + 2}.$$

Payoff

For the **first part** of $x \in (0, A_k)$, the payoff is (use relation of C_i)

$$P_1 = \int_0^{A_k} C_i - (R_i + 1)(1 - C_i)\, dx = \int_0^{A_k} C_k - (R_k + 1)(1 - C_k)\, dx = A_k(2C_k - 1).$$

For the **second part** of $x \in (A_k, B_k)$, the payoff is

$$P_2 = \int_{A_k}^{B_k} 2x - 1\, dx = B_k^2 - B_k - A_k^2 + A_k.$$

With this relation of C_i, the payoff for the **last part** at each $x \in (B_k, 1)$ becomes

$$\frac{R + 2C_k}{R + 2} + (R + 1)\left(x - \frac{R + 2C_k}{R + 2}\right) - (R + 1)(1 - x).$$

Hence, for a given x, the optimal value of raise satisfies the relation:

$$x = \frac{R^2 + 4R + 2C_k + 2}{(R+2)^2}. \tag{1}$$

Then the whole payoff for the interval $x \in (B_k, 1)$ becomes

$$\begin{aligned} P_3 &= \int_{B_k}^{1} \frac{R + 2C_k}{R+2} + (R+1)\left(x - \frac{R+2C_k}{R+2}\right) - (R+1)(1-x)\,dx \\ &= \int_0^{\infty} \frac{(3-2C_k)R^2 + 4R + 4C_k}{(R+2)^2} \cdot \frac{4(1-C_k)}{(R+2)^3}\,dR \\ &= \frac{(5+C_k)(1-C_k)}{12}. \end{aligned}$$

We stress again that $A_k = C_k$ and $B_k = x_{\{R=0\}} = \dfrac{C_k + 1}{2}$.

Hence the total payoff is

$$
\begin{aligned}
P &= P_1 + P_2 + P_3 \\
&= A_k(2C_k - 1) + B_k^2 - B_k - A_k^2 + A_k + \frac{(5 + C_k)(1 - C_k)}{12} \\
&= \frac{7}{6}C_k^2 - \frac{C_k}{3} + \frac{1}{6}.
\end{aligned}
$$

P has an optimal value at $C_k = \dfrac{1}{7}$ with value $\dfrac{1}{7}$.

Note: $A_k = C_k = 1/7$ and $B_k = 4/7$.

This would be the end to this problem. However, we can also figure out the amount of bet when $x < A_k = 1/7$ as following:

For each i, $1 \leq i \leq k-1$,

$$\frac{\partial P}{\partial C_i} = 0 \implies [1 + (R_i + 1)](A_i - A_{i-1}) = -[1 - (R_i + 1)](B_{i-1} - B_i).$$

That is

$$A_i - A_{i-1} = \frac{R_i}{R_i + 2}(B_{i-1} - B_i).$$

Sum i from 1 to s and change summation to integration,

$$A_s = \int_1^{B_s} \frac{R}{R+2}\, dB = \int_{R'}^{\infty} \frac{R}{R+2} \cdot \frac{24}{7(R+2)^3}\, dR = \frac{1}{7} - \frac{R'^2(R'+6)}{7(R'+2)^3},$$

where the relation of B and R follows from (1).

Summarize of the Strategies

Player 1

Case 1: For a given card $x < A_k = 1/7$, Player 1 should bet with amount R that satisfies the relation

$$x = \frac{1}{7} - \frac{R^2(R+6)}{7(R+2)^3} = \frac{4(3R+2)}{7(R+2)^3}.$$

Case 2: For a given card $1/7 = A_k < x < B_k = 4/7$, Player 1 should check.

Case 3: For a given card $4/7 = B_k < x$, Player 1 should bet with amount R that satisfies the relation

$$x = \frac{R^2 + 4R + 2C_k + 2}{(R+2)^2} = \frac{7R^2 + 28R + 16}{7(R+2)^2}.$$

Player 2

In respond to amount of raise $R > 0$ of player 1, player 2 should fold if his/her card $y < C'$ and call if $y > C'$, where

$$C' = \frac{R + 2C_k}{R + 2} = \frac{7R + 2}{7R + 14}.$$

These results *almost* agree with solution of Donald J. Newman [2] on the second page. The difference is the raise amounts on the interval when player 1 bluffs. Both solutions are valid.

References

[1] S. Bargmann. *On the theory of games of strategy*, 40:13-42, 1959. English translation of J. von Neumann first poker paper.

[2] D. J. Newman. *A model for "real" poker*, Operations Research, 7(5):557-560, 1959.

[3] J. von Neumann and O.Morgenstern. *Theory of Games and Economic Behavior*, Princeton University Press, 1944.