Apéry Miracle

Thotsaporn Thanatipanonda

Mahidol University International College,
Nakornpathom, Thailand

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Background

Figure: Roger Apéry

Figure: His famous work in 1979

Roger Apéry was born in Rouen in 1916 to a French mother and Greek father. After studies at the cole Normale Suprieure (interrupted by a year as prisoner of war during World War II) he was appointed Lecturer at Rennes. In 1949 he was appointed Professor at the University of Caen where he remained until his retirement. In 1979 he published an unexpected proof of the irrationality of $\zeta(3)$

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- Proof of irrationality of $\sqrt{2}$ by contradiction.

How it should have been proved for $\sqrt{2}$ to be irrational

To rigorously proof that $\sqrt{2}$ is irrational.
First consider

$$y := 2 + 1/(2 + 1/(2 + 1/(2 + 1/(2 + ...$$

By using **self-similarly,** deduce the equation.

$$y = 2 + 1/y$$

Solving the quadratic equation to get $y = 1 + \sqrt{2}$. Hence, the infinite fraction indeed equals $1 + \sqrt{2}$.
Therefore $1 + \sqrt{2}$ and $\sqrt{2}$ are indeed irrational.

This proof is much better because ...

- $$\frac{p_n}{q_n} := [2, 2, 2, \ldots, 2] \qquad \text{, repeated } n \text{ times.}$$

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  - Furthermore, the error, $\xi - \frac{p_n}{q_n}$ can be easily estimated:

    $$\left|\xi - \frac{p_n}{q_n}\right| = \left|\sum_{i=n+1}^{\infty} \frac{(-1)^i}{q_i q_{i-1}}\right| \leq \frac{C}{q_n^2} \quad ,$$

    for some easily computable constant, C.

Dirichlet's Approximation Theorem

- This is a fundamental result in Diophantine approximation , showing that any real number has a sequence of good rational approximations (Dirichlet):

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- However if ξ is a rational a/b. If $\xi \neq r$, then

$$|\xi - r| = \left|\frac{a}{b} - \frac{p}{q}\right| = \frac{|bp - aq|}{bq} \geq \frac{1}{bq},$$

So equation (1) involves $q < b$. There are therefore only a finite number of solutions of equation (1).

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- This leads to

Theorem (irrationality criterion)

If there is a $\delta > 1$ *and an infinite sequence of distinct* $\{p_n/q_n\}$ *of rational numbers such that* $\xi \neq p_n/q_n$ *and*

$$\left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{\delta}}, \qquad n = 1, 2, \ldots$$

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- Apéry's original proof was based on this well known irrationality criterion.

Apéry's Proof

- Apery defined a sequence $c_{n,k}$ by

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}}.$$

He then defined two more sequences a_n and b_n that, roughly, have the quotient $c_{n,k}$. These sequences were

$$a_n = \sum_{k=0}^{n} c_{n,k} \binom{n}{k}^2 \binom{n+k}{k}^2$$

and

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

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- **Fact 1:**
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \zeta(3).$$
- **Fact 2:** Both a_n and b_n satisfy the same recurrence relation:

$$n^3 u_n - (34n^3 - 51n^2 + 27n - 5) u_{n-1} + (n-1)^3 u_{n-2} = 0, \qquad n \geq 2.$$

with the initial conditions $u_0 = 0$ and $u_1 = 6$ for a_n and $u_0 = 1$ and $u_1 = 5$ for b_n.

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$$\left|\zeta(3) - \frac{a_n}{b_n}\right| = O(b_n^{-2}).$$
- **Fact 4:** $b_n \approx c \cdot (1+\sqrt{2})^{4n}$.
- **Fact 5:** Let $p_n = 2\,lcm(1,2,3,\ldots,n)^3 a_n$ and $q_n = 2\,lcm(1,2,3,\ldots,n)^3 b_n$ then

$$\left|\zeta(3) - \frac{p_n}{q_n}\right| = C/q_n^{\delta} \quad \text{where } \delta = \frac{2\log\alpha}{\log\alpha + 3} \approx 1.080529\ldots > 1.$$

Then by the irrationality criterion, we conclude that $\zeta(3)$ is irrational.

Searching for Miracle

The Three kinds of Apéry Miracles

- A minor Apéry miracle is when

$$\alpha := \lim_{n\to\infty} \frac{p(n)}{q(n)}$$

  equals a well known constant, expressible in terms of π, e, γ, etc.

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- A super Apéry miracle is when, in addition, the constant α has not yet been proven to be irrational, making you famous.

It's All Come Down to Experimental Mathematics

Mathematics is an experimental science.
There is a room to search for another miracle (?)

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- Based on this idea, T. Rivoal, in 2000, showed that there are infinitely many irrational numbers among $\zeta(3), \zeta(5), \zeta(7), \ldots$.
- Then Zudilin, in 2001, showed that at least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.