SYMBOLIC-COMPUTATIONAL METHODS IN COMBINATORIAL GAME THEORY AND RAMSEY THEORY

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ABSTRACT OF THE DISSERTATION

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This thesis is a contribution to the emerging field of *experimental rigorous mathematics*, where one uses symbolic computation to conjecture proof-plans, and then proceeds to verify the conjectured proofs rigorously. The proved results, in addition to their independent interest, should also be viewed as **case studies** in this budding methodology. We now proceed to described the specific results presented in this dissertation.

We first develop a finite-state automata approach, implemented in a Maple package `ToadsAndFrogs`, for conjecturing, and then rigorously proving, values for large families of positions in Richard Guy's combinatorial game "Toads and Frogs".

In particular, we prove conjectures of Jeff Erickson. We also discuss the values of all positions with exactly one $\square$, $T^a\square\square F^a$, $T^a\square\square\square FFF$, $T^a\square\square F^b$, $T^a\square\square\square F^b$.

We next consider the generalized chess problem of checkmating a king with a king and a rook on an $m \times n$ board at a specific starting position. We analyze the fastest way to checkmate.

We also consider a problem posed by Ronald Graham about the minimum number, over all 2-colorings of $[1, n]$, of generalized so-called Schur triples, i.e. monochromatic triples of the form $(x, y, x + ay)$ $a \geq 1$. (The case $a = 1$ corresponds to the classical Schur triples). In addition to giving a completely new proof of the already known case of $a = 1$, we show that the minimum number of such triples is at most $\frac{n^2}{2a(a^2+2a+3)} + O(n)$ when $a \geq 2$. We also find a new upper bound for the minimum number, over all r-colorings of $[1, n]$, of monochromatic Schur triples, for $r \geq 3$.

Finally, in yet a different direction, we find closed-form expressions for the second moment of the random variable "number of monochromatic Schur triples" defined on the sample space of all r-colorings of the first n integers, and second and even higher moments for the number of monochromatic complete graphs K_k in K_n. In addition to their considerable independent interest, these formulas would hopefully be instrumental in improving the extremely weak known lower bounds for the asymptotics of Ramsey number.

Acknowledgements

I want to thank my advisor, Dr. Doron Zeilberger, for supporting me all these years in graduate school. I may not have been the best student, but he helped and motivated me all along. I learned so much mathematics, symbolic programming, and philosophy of learning and teaching, from him. He is not only a great mathematician but also a very kind person, and he makes mathematics fun.

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I also wish to thank Bruce Landman and Aaron Robertson for their beautiful book *Ramsey Theory on the Integers*, that explained Ramsey theory so well. I really enjoyed reading their book. Many thanks are due to Yoni Berkowitz for helpful discussions, and to Thomas Robinson for discussions on the moments of Ramsey graphs.

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I owe many debts of gratitude to my many students for their patience and friendship. They made me love teaching.

The defense committee members, Prof. Vladimir Ratakh, Dr. Neil Sloane and Prof. Michael Saks are hereby thanked for their valuable time and priceless comments, that dramatically improved the readability of this thesis.

Last but not least, I want to thank my parents for everything they did.

Dedication

To parents who have always supported me

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Chapter 1

Introduction: Summary and Background Stories

During my years in graduate school, I learned the philosophy and methodology of using computers in mathematical research from my advisor, Professor Doron Zeilberger.

In my opinion, it is not very important in what mathematical area I am working on, since the *experience* gained in doing computer-assisted and computer-generated research in one area are likely to be transferable to other areas.

Richard Hamming (1915-1998), a great applied mathematician, said "the purpose of computing is insight, not numbers". Insight leads to understanding. Computation gives me insight in two different ways. First the *act* of programming makes me understand the problem much better, and at a deeper level; second, the *output* often leads to further understanding. Once we collect all the information, we see the *big picture* without worrying about the details of the computations. I like to solve challenging problems, and it is always the case that computer programming helps the computational parts go smoother.

We can divide this thesis to 4 main independent parts.

1) The Combinatorial Game *Toads and Frogs*.

2) Generalized Chess endgame problems .

3) Ramsey Theory, in particular, the minimum number of generalized monochromatic Schur triples in r-colorings the first n integers.

4) Symbolic Moment Calculus and its applications.

Let me now give some background, and future plans, for each of these problems.

1.1 The Combinatorial Game *Toads and Frogs*

The modern theory of combinatorial games was developed by J.Conway, E.Berlekamp, and R.Guy, who wrote the classic book *Winning Ways*, that mostly deals with *partizan* games, and by Aviezri Fraenkel and his many students, who study *impartial* games.

The combinatorial game *Toads and Frogs* was introduced for the first time by Richard Guy in [1]. In 1996, Jeff Erickson [4] performed a more detailed study, and discovered more patterns. He made six conjectures at the end of his paper. In 2000, Jesse Hull proved one of his conjectures. He proved expilicit formulas for the game-values (in the sense of Conway) for certain infinite families of game-positions. These values imply that *Toads and Frogs* is NP-hard, in general. All the other five conjectures were still open, and four of them are settled in this thesis.

At the beginning, I wrote a program in Maple to calculate values of *specific* "Toads and Frogs" positions. Using this data, I developed (in collaboration with Zeilberger) an algorithm called *symbolic finite state method*, that allowed us to perform automated proofs of explicit expressions for the values of many infinite families of game-positions. This algorithm was fully implemented in Maple.

In Chapter 2, I will introduce the *symbolic finite state method* and illustrate it with examples. In Chapter 3, I describe how to make a database of the values for each such class of "Toads and Frogs" position. These values all come from the symbolic finite state method. In Chapter 4, we explore the more general patterns of positions that seemed beyond the scope of the (fully computer-generated) finite state method, and that, at least for now, required human intervention, and are merely *computer-assisted*. We present new tables, formulate further conjectures and talk about possible future work. In Chapter 5, we prove the values of positions with even more general patterns than those found in Chapter 4. These positions could not (yet) be proved by computer program, and were done, in part, by hand.

Combining computer and human efforts, we settled four of Erickson's conjectures: three are positive and one is negative. The last conjecture is still open.

In the future, we hope to apply the finite state method to other combinatorial games, especially *the rook endgame problem* which we will talk about in chapter 6.

1.2 Problems in Generalized Chess endgame problems

Chess was my favorite hobby when I was in college. My dad does not like to see me play chess, since he thinks that it is a waste of time, but my advisor is very interested in chess endgame problems, since he believes that they have an interesting mathematical structure, so I was fortunate to combine "business" with "pleasure" in the present project.

When we started the project, he bought me the classic chess book [2] written by Capablanca, the third world chess champion. The first diagram in that book depicts an endgame problem featuring the two kings and one white rook, and the problem is to find the smallest number of moves for White to checkmate for a given starting position P. We call that number $C(P)$.

The rook problem is original and elementary. We wrote a program called *Rook* to find $C(P)$. We found an improvement from the moves Capablanca suggested in his book. We also investigated the rook endgame problem on an $m \times n$ board instead of the usual 8×8 board.

I found a way of applying the symbolic finite-state method to the rook problem to solve for $C(P)$ for all positions on a $k \times n$ board where $k \geq 3$ is fixed. It is also of interest to find out whether $C(P) \leq m+n$ for every position P of rook problem on an $m \times n$ board for all $m \geq 3$ and $n \geq 3$ using the finite state method.

1.3 Problems on the minimal number of monochromatic Schur Triples

In 1916, I. Schur [11] proved that for every $r \geq 2$, there exists a least integer $n = S(r)$, such that for every r-coloring of $[1, n]$, there exists a monochromatic solution to $x+y = z$. The integers $S(r)$ are called Schur numbers. For example $S(2) = 5$. On the interval [1,4], you can color the integers with [r,b,b,r] with no monochromatic Schur triples. But on [1,5], you will always have at least one monochromatic Schur Triple for each and every one of the 2^5 ways of coloring the first five integers.

In 1995, Graham, Rodl and Rucinski proposed the following problem: Find the asymptotic minimum number of monochromatic solutions to the equation $x + y = z$ amongst all 2-coloring of $[1, n]$. The problem was solved independently solved in [9] and [10]. Another proof was given later in [3]. The answer is $\frac{n^2}{22} + O(n)$.

Shortly after, Graham generalized the problem and asked for the asymptotic minimum number of monochromatic solutions to the equation $x + ay = z, a \geq 2$ amongst all 2-colorings of $[1, n]$. An analogous problem is discussed in [8], where the equation is $x + y = 2z$, describing a 3-term arithmetic progression.

In this chapter, we give a novel proof, using completely new ideas, of the original problem. We also find a new algorithm to find a good upper bound for the original problem with r-colorings rather than just 2-colorings, as well as for Graham's generalized problem. It is a "greedy" type algorithm, using calculus. We conjecture that these upper bounds are the actual minimum values. Finally, we managed to find two new lower bounds when $a = 2, 3$ for the generalize problem.

I am also interested in analogous problems for graphs. One such problem can be stated as follows. Find the asymptotic minimum number of monochromatic K_k of any r-edge-coloring of K_n, where $k \geq 3$ and $r \geq 2$ are fixed. The answer is known only for $(k, r) = (3, 2)$. The minimum turns out to be the same as the average which turns out to be $\frac{n^3}{24} + O(n^2)$. I hope to work on this problem in the near future.

1.4 Symbolic Moment Calculus and its application

In this chapter, I calculated higher moments of random variables associated with two different combinatorial objects. From my experience working on these problems, going from one moment to another requires a lot more computations. Most of the time, computing the third moment is very hard. Many mathematicians do not like this type of problem because of their difficulty and the long, tedious answer they get. The original work can be found in [5]. Zeilberger pioneered symbolic computational methods for computing higher moments for interesting random variables. His work about computing higher moments can be found in [12] and [13].

The first such random variable we considered is the number of monochromatic Schur Triples defined in the sample space of all r coloring of $[1, n]$. We managed to compute the first and second moments, exactly. Another random variable considered is the number of monochromatic K_k in r-edge-colorings of the complete graph K_n. We found formulas for moments in terms of certain multi-sums. However to write out the explicit formula from these sums is still hard. We wrote a program to compute explicit formulas up to the fifth moment. The input is the numeric k, and the output is the formula for each moment in terms of n (the number of vertices) and r (the number of colors).

In a ground-breaking work, Paul Erdös used the first moment, (alias expectation), to show that $\lim inf_{k \to \infty} R(k, k)^{\frac{1}{k}} \geq \sqrt{2}$. We use an idea of Zeilberger in [13], that uses the generalized Principle of Inclusion-Exclusion (PIE) with higher moments in the hope of improving the lower bound of $\lim inf_{k\to\infty} R(k,k)^{\frac{1}{k}}$. The second part of chapter 8 expands the details of Zeilberger's idea. We realize that this is a "long-shot", and still very exploratory, but we believe that the problem is so interesting that is worth exploring.

The problem about improving the lower bound of $\lim inf_{k\to\infty} R(k,k)^{\frac{1}{k}}$ is a very famous problem. The lower bound has not been improved since Erdos first introduced the idea of the probabilistic method more than 60 years ago.

I like to compare this problem with the Four Color Theorem in graph theory. For a long time people tried to find a short, elegant proof, without success. At the end, many people realized that they have to get their hands dirty by working on details which require lots of case analysis. I have the feeling that this problem might end up the same way. Extensive computations are required in order to gain more information that we need to improve the lower bound on $\lim inf_{k\to\infty} R(k,k)^{\frac{1}{k}}$.

Chapter 2

Toads and Frogs (Symbolic Finite-State Approach)

2.1 Introduction

The game *Toads and Frogs*, invented by Richard Guy, is extensively discussed in "Winning Ways" [1], the famous classic by Elwyn Berlekamp, John Conway, and Richard Guy, that is the *bible* of combinatorial game theory.

This game got so much coverage because of the simplicity and elegance of its rules, the beauty of its analysis, and as an example of a combinatorial game whose positions do not always have values that are numbers.

The game is played on a $1 \times n$ strip with either Toad(T) , Frog(F) or $\square$ on the squares. Left plays T and Right plays F. T may move to the immediate square on its right, if it happens to be empty, and F moves to the next empty square on the left, if it is empty. If T and F are next to each other, they have an option to jump over one another, in their designated directions, provided they lend on an empty square. (See [1], page 14).

In symbols: the following moves are legal for Toad:

$\ldots \mathrm{T}\square \ldots \;\rightarrow\; \ldots \square\mathrm{T} \ldots,$

$\ldots \mathrm{TF}\square \ldots \;\rightarrow\; \ldots \square\mathrm{FT} \ldots \quad ,$

and the following moves are legal for Frog:

$\ldots \square \mathrm{F} \ldots \quad \rightarrow \quad \ldots \mathrm{F} \square \ldots,$

$\ldots \square \mathrm{TF} \ldots \quad \rightarrow \quad \ldots \mathrm{FT} \square \ldots \quad .$

Already in "Winning Ways" [1], there is some analysis of Toads and Frogs positions, but on *specific*, small boards, such as TTT□FF. In 1996, Jeff Erickson [4] analyzed more general positions. At the end he made five conjectures about the values of some families of positions. All of them are "starting" positions (i.e. positions where all T's are rightmost and all F's are leftmost).

To be able to understand this chapter, readers need some knowledge of combinatorial game theory, that can be found in [1]. In particular, readers should be familiar with the notion of *value* of a game. Recall that values are not always numbers (not even surreal ones).

Let's recall the *bypass reversible move rule*, the *dominated options rule* (see [1] page 62-64) and Erickson's *Terminal Toads Theorem* (see [4]).

Bypassing right's reversible move rule

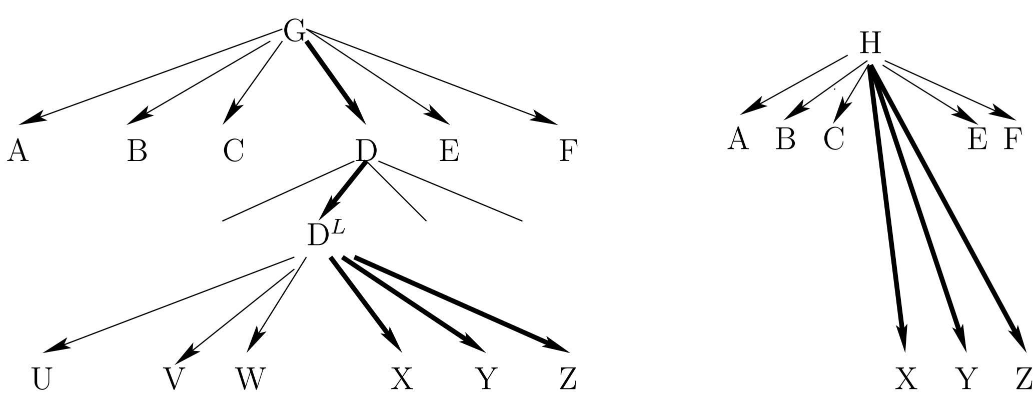


Figure 2.1: The Bypassing reversible move rule.

$G = H$ if $D^L \geq G$.

The Dominated options Rule

Let $G = \{A, B, C, ... \mid D, E, F, ..\}$.
If $A \geq B$ and $D \geq E$ then $G = \{A, C, ... \mid E, F, ..\}$.

The Terminal Toads Theorem: Let X be any position. Then
XT$\square^n$ = X$\square^n + n$.

The only notation we use is $*$ $(= \{0 \mid 0\})$. We will not use any shorthand notation like $\uparrow$, $\Uparrow$, etc.

Next, we will explain the method through examples, and describe how to implement the method when applied to certain classes of positions. Finally, we discuss a new conjecture and possible future work.

Everything is fully implemented in a Maple package, `ToadsAndFrogs`, written by the author, available from website.

2.2 A Symbolic Finite-State Method

We define two classes of positions:

Class **A**: All the positions that have a *fixed* number of occurrences of $\square$ and F, but a *variable* (symbolic) number of T's in-between the $\square$'s and F's.

class **B**: All the positions that have a *fixed* number of occurrences of T's and F's, but a *variable* (symbolic) number of $\square$'s in-between the T's and F's.

Aij := the class in which we have exactly i occurrences of $\square$ and exactly j occurrences of F.

Bij := the class in which we have exactly i occurrences of T and exactly j occurrences of F.

For any *specific* position, we can always compute its value, by using the recursive definition of the value. But this is mere *number-crunching*. After collecting enough data, and examining it, if we are lucky, we (or rather our computers) can detect a uniform *pattern*, and **conjecture** an **explicit** formula for the values of the studied family, in terms of the symbolic parameters. Once conjectured, these conjectured explicit expressions can be proved by induction on the symbolic parameters. The beauty and novelty of our approach is that everything is done **automatically**. First the *conjecturing* parts, but more dramatically, the *proving* part. We teach the computer how to conjecture, by looking for general patterns, and then how to use induction in order to prove its own conjectures.

This *activity* of **computer-generated** mathematics is in sharp contrast to the traditional approach of [2], that merely uses the computer as a calculator, to generate numerical data, and everything else, the conjecturing, and the proving (when feasible) is done by humans.

We believe that the present methodology is of potential use in many other branches of mathematics, and "Toads and Frogs" is but an instructive arena for presenting a general approach for computer-generated research.

When we analyze each class of positions, we are naturally lead, by the recursive definition of the *value* (of a game), to other classes of positions. Luckily, at least in all the cases encountered so far, there are always a **finite** number of different classes, that we can name "symbolic states". If the (symbolic) value of each "state" in the class is conjectured to have a (symbolic) explicit expression, then we can prove the truth of *all* these conjectures **all at once** by applying induction on the recurrence relations. Note that in order for this to work we need to conjecture explicit expressions for **all** the states, so we usually get much more than we bargained for.

We will demonstrate the method with the two simplest nontrivial classes: A11 and B11.

First example: Type A11: one □ and one F

Let $f(a,b)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}$.
Let $g(a)$ be the value of $\mathrm{T}^a\mathrm{F}\square$.

Here, of course, T^a means T repeated a times, so the 'game' $f(a,b)$, for example, stands for a doubly-infinite set of starting positions.

Recurrences:
Note that if any parameter of the function is negative then it return NULL.

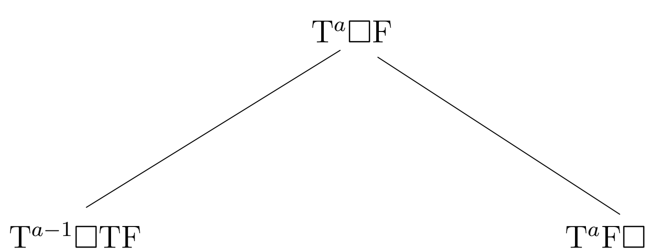


Figure 2.2: Recurrence for $f(a,0), a \geq 0$.

$$f(a,0) = \{f(a-1,1) \mid g(a)\}, a \geq 0.$$

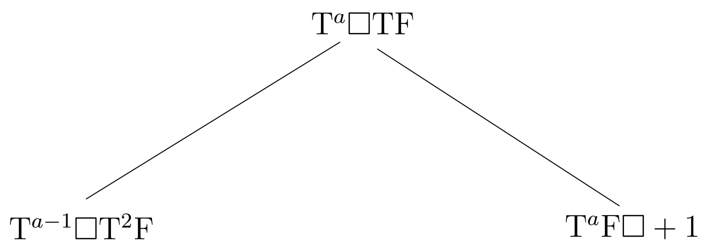


Figure 2.3: Recurrence for $f(a, 1), a \geq 0$.

$$f(a, 1) = \{f(a-1, 2) \mid g(a) + 1\}, a \geq 0.$$

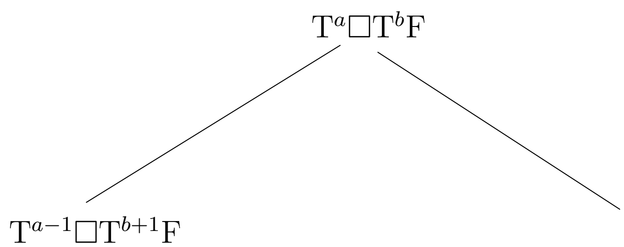


Figure 2.4: Recurrence for $f(a, b), a \geq 0, b \geq 2$.

$$f(a, b) = \{f(a-1, b+1) \mid \ \}, a \geq 0, b \geq 2.$$

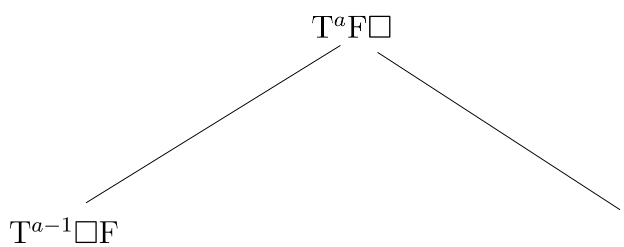


Figure 2.5: Recurrence for $g(a), a \geq 0$.

$$g(a) = \{f(a-1, 0) \mid \ \}, a \geq 0.$$

The above recurrences can be easily used to crank out *numerical data* for small (and not so small) values of a and b. Then the computer *automatically* makes the following *symbolic* conjectures.

Conjectures:

$$\begin{aligned}
f(0,0) &= -1. \\
f(a,0) &= \{\{a-2 \mid 1\} \mid 0\} \quad , \quad a \geq 1. \\
f(a,1) &= \{a-1 \mid 1\} \quad , \quad a \geq 0. \\
f(a,b) &= a \quad , \quad a \geq 0, \quad b \geq 2. \\
g(a) &= 0 \quad , \quad a \geq 0.
\end{aligned}$$

Once conjectured, the proof is routine, and also can (and was!) done by computer. One checks the obvious initial conditions and verifies that the above expressions satisfy the above *defining* relations. Indeed, the computer easily verifies that

$$\begin{aligned}
f(a,0) &= \{f(a-1,1) \mid g(a)\} = \{\{a-2 \mid 1\} \mid 0\} \quad , \quad a \geq 1. \\
f(0,1) &= \{\ \mid g(0)+1\} = \{\ \mid 1\} = 0 = \{-1 \mid 1\}. \\
f(a,1) &= \{f(a-1,2) \mid g(a)+1\} = \{a-1 \mid 1\} \quad , \quad a \geq 1. \\
f(0,b) &= \{\ \mid\ \} = 0 \quad , \quad b \geq 2. \\
f(a,b) &= \{f(a-1,b+1) \mid\ \} = \{a-1 \mid\ \} = a \quad , \quad a \geq 1, b \geq 2. \\
g(0) &= \{\ \mid\ \} = 0. \\
g(1) &= \{f(0,0) \mid\ \} = \{-1 \mid\ \} = 0. \\
g(a) &= \{f(a-1,0) \mid\ \} = \{\{\{a-3 \mid 1\} \mid 0\} \mid\ \} \\
&= \{\ \mid\ \} \text{ (!! by bypass reversible move rule)} = 0 \quad , \quad a \geq 2.
\end{aligned}$$

Note that the above values for $f(a,0)$ $(a \geq 1)$ agree with the case $b = 1$ of Theorem 5.2 of [2].

Second Example: Type B11: one T and one F.

Let $f(a,b,c) := \square^a \mathrm{T} \square^b \mathrm{F} \square^c$.

Now we have a *three*- parameter family!

Initial Conditions and Recurrences:

$$
\begin{aligned}
f(0,0,0) &= \{\ \mid\ \}. \\
f(a,0,0) &= \{\ \mid (-a+1)+1\} = \{\ \mid -a+2\} \quad , \quad a \geq 1. \\
f(0,0,c) &= \{(c-1)-1 \mid\ \} = \{c-2 \mid\ \} \quad , \quad c \geq 1. \\
f(a,0,c) &= \{c-a-2 \mid c-a+2\} \quad , \quad a \geq 1, c \geq 1.
\end{aligned}
$$

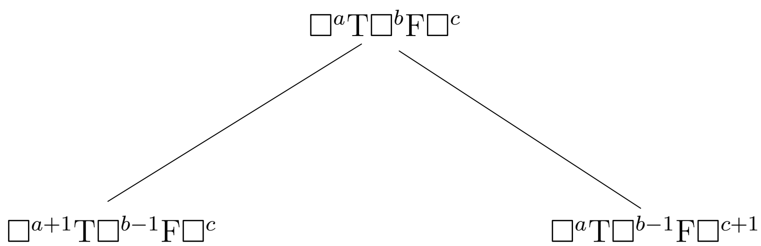


Figure 2.6: Recurrence of $f(a,b,c), a \geq 0, c \geq 0, b \geq 1$.

$$f(a,b,c) = \{f(a+1,b-1,c) \mid f(a,b-1,c+1)\}, \quad a \geq 0, c \geq 0, b \geq 1.$$

By using these recurrences *numerically*, the computer cranks out enough data, that enables it to make the following

Conjecture:

$$
\begin{aligned}
f(a,b,c) &= \{c-a-2 \mid c-a+2\} \quad , \quad a \geq 0, c \geq 0, b \text{ is even.} \\
f(a,b,c) &= \{\{c-a-3 \mid c-a+1\} \mid \{c-a-1 \mid c-a+3\}\} \quad , \quad a \geq 0, c \geq 0, \ b \text{ is odd .}
\end{aligned}
$$

Proof: by induction: on b.

Base case: $b = 0$

We have

$$
\begin{aligned}
f(0,0,0) &= 0 = \{-2 \mid 2\}. \\
f(a,0,0) &= \{\ \mid -a+2\} = \{-a-2 \mid -a+2\}, \quad a \geq 1. \\
f(0,0,c) &= \{c-2 \mid\ \} = \{c-2 \mid c+2\}, \quad c \geq 1. \\
f(a,0,c) &= \{c-a-2 \mid c-a+2\}, \quad a \geq 1, c \geq 1.
\end{aligned}
$$

Induction step on b

Case 1) b is even and $b \neq 0$:

$$\begin{aligned} f(a,b,c) &= \{f(a+1,b-1,c) \mid f(a,b-1,c+1)\}, \quad a \geq 0, c \geq 0. \\ &= \{\{c-a-4 \mid c-a\} \mid \{c-a-2 \mid c-a+2\}\} \\ &\quad \mid \{\{c-a-2 \mid c-a+2\} \mid \{c-a \mid c-a+4\}\}\}. \\ &= \{c-a-2 \mid c-a+2\}. \end{aligned}$$

Case 2) b is odd

$$\begin{aligned} f(a,b,c) &= \{f(a+1,b-1,c) \mid f(a,b-1,c+1)\} \qquad , a \geq 0, c \geq 0. \\ &= \{\{c-a-3 \mid c-a+1\} \mid \{c-a-1 \mid c-a+3\}\}. \end{aligned}$$

The second example is related to the results of Erickson [4] as follows. The case $b = 0$ is Lemma 4.1 of [4], while the case $a = 0, c = 0$ coincides with the case $a = 1$ of Theorem 5.2. Note that we need the extra elbow-room of a three-parameter family to enable the inductive argument.

2.3 How far can the symbolic finite state method go?

As we mentioned in the previous section, the finite state method works perfectly well when the value of every position in the class has a discernible pattern. This seems to be the case for class A. We wrote a computer program in Maple to first *calculate*, then *conjecture*, and finally *prove*, the values of general positions in class A automatically. The program now works for positions with any fixed number of □'s and with one Frog. For the class where we have more than one Frog, it is harder to find conjectures, for humans, and *even* for computers. We conjectured some classes with two Frogs(A12, A22, A32) by hand and put it in the computer program to prove the conjectures.

The list of the results for the classes A11, A21, A31, A41, A51, A12, A22, A32 can be found in the next chapter.

As a very special case of our results for the class A32, we get a proof of Erickson's [4] conjecture 2, that claims that the value of $\mathrm{T}^a\Box\Box\Box\mathrm{FF}$ is $\{a-2 \mid a-2\}$, ($a \geq 2$).

In the next chapter, we discusses the value of *any* position with one $\Box$ and *any* number of Toads and Frogs (Therefore we are done with class $A1n$, $n \geq 1$). This general class with one $\Box$ is the only general class we are able to figure out the patterns for.

We now turn our attention to class B. We solved class B11 in the previous section. For B21: TTF, we already have a difficulty. The formulas in this class are long and hard to find in a canonical form. We will discuss this in the next section.

2.4 A Conjecture and Future Work

Conjecture:

1) We always have "nice compact" formulas for every position in class A.

Future Work:

1) Implement the symbolic finite state method for the class B21.

2) We have seen systems of recurrence relations arising naturally in each class. We solved the recurrences by "guessing" (automatically, of course) the answers (using predefined *ansatzes*) and then proving them by induction. It would be interesting to develop general algorithms for systematically solving the recurrences, without the need for "guessing".

2.5 On the difficulty of class B21: TTF

B21: TTF

$f(a,b,c,d) := \Box^a\mathrm{T}\Box^b\mathrm{T}\Box^c\mathrm{F}\Box^d$.

$g(a,b,c) := \square^a \mathrm{T} \square^b \mathrm{F} \square^c.$

We already knew the solution of g since it is exactly B11.

We can now focus on f.

Recurrences:

$$\begin{aligned}
f(a,0,0,0) &= \{\ \mid\ \} = 0. \\
f(a,0,0,d) &= \{g(a,1,d-1)+d-1 \mid\ \} \\
&= \{\{\{d-a-4 \mid d-a\} \mid \{d-a-2 \mid d-a+2\}\} \mid\ \} \\
&\quad ,\ a \geq 0,\quad d \geq 1. \\
f(a,b,0,0) &= \{f(a+1,b-1,0,0) \mid g(a,b-1,1)+1\} \\
&\quad ,\ a \geq 0, b \geq 1. \\
f(a,b,0,d) &= \{f(a+1,b-1,0,d),\quad g(a,b+1,d-1)+d-1 \mid g(a,b-1,d+1)+d+1\} \\
&\quad ,\ a \geq 0, b \geq 1, d \geq 1. \\
f(a,b,c,d) &= \{f(a+1,b-1,c,d),\quad f(a,b+1,c-1,d) \mid f(a,b,c-1,d+1)\} \\
&\quad ,\ a \geq 0, b \geq 0, c \geq 1, d \geq 0.
\end{aligned}$$

Note: $f(a,0,0,d)$ has been discussed before as lemma 4.3 by Erickson.

A nice formula for $f(a,b,0,0)$.

For b =1:

$$\begin{aligned}
f(a,1,0,0) &= \{f(a+1,0,0,0) \mid g(a,0,1)+1\} \\
&= \{0 \mid \{-1-a \mid 3-a\}+1\} \\
&= \begin{cases} \frac{1}{2} & \text{, a = 0,1} \\ \{0 \mid 3-a\} & \text{, a} \geq 2 \end{cases}
\end{aligned}$$

For b $\geq$ 2 and b is even:

$$\begin{aligned}
f(a,b,0,0) &= \begin{cases} 1 & \text{, a = 0} \\ -a+2 & \text{, a } \geq 1. \end{cases} \\
&= \{\ \mid a\} - a + 2.
\end{aligned}$$

For b $\geq$ 2 and b is odd.

$$f(a,b,0,0) = \begin{cases} \{1 \mid 1\} & ,a=0 \\ \frac{1}{2} & ,a=1 \\ -a+2 & ,a \geq 2. \end{cases}$$

However for $f(a,b,0,d), a \geq 0, b \geq 1, d \geq 1$, the formulas get longer and longer and we started to lose track of them, and consequently failed to find formulas in this case. It should be possible to write Maple code specifically to find a pattern for the values of positions in class B. The authors expect the formulas in other classes of type B (for example B22: TTFF) to be even more complicated than B21, since it has to build up from B21.

It appears that the positions in class B have periodicity and they need more care to formulate the right conjectures.

2.6 About the program

Our Symbolic Finite-State Method was implemented in Maple. We first wrote a program to recursively calculate the values of games. Then we improved the program by making use of the symbolic computation capability of Maple, to formulate conjectures, and prove the values of game-positions. The whole proof process was completely automated. Below is the short description of the program. See the web site for complete details of the program.

ToFr

Input: the specific position of the game.

Output: the value of the game in canonical form.

SVG

Input: the value of the game, could be symbolic.

Output: the value of the game in canonical form.

Note: This program can also be used for other combinatorial games.

MainConj

Input: number of $\square$ and number of F.

Output: The list of conjectures.

Prove

Input: number of $\square$ and number of F.

Output: the values of all of the positions in this class.

The program currently only works for one Frog with any fixed number of $\square$. With more than one Frog, it gets harder to find conjectures. But one could find conjectures by hand and feed them to the subfunctions in Prove. The program can help verify such humanly-made conjectures.

Obviously, there is still a lot of work to be done, but let's remember that

" **Every great artwork always starts from a rough draft**".

Chapter 3

Library values of Toads and Frogs

3.1 Introduction

We present here the values of Toads and Frogs as an implementation of the finite state method introduced in the previous chapter.

All the values in that chapter have already been proved. For the class with one Frog, $Ai1, \quad i = 1, 2, 3, 4, 5$, we have an automated program to conjecture and prove everything automatically. For the class with two frogs, $Ai2, \quad i = 1, 2, 3$, we also have to use human ingenuity to formulate conjectures , but use the computer program that we wrote to prove these conjectures. For the class with one blank, $A1i$ we outlined the fast algorithm to compute the values of positions. We then compute the explicit values of $A1i, \quad i = 1, 2, 3$. For the class $B11$, we already proved the values by hand, as an example, in the previous chapter. We also mention it here.

3.2 Results for classes with one frog.

ClassA11: □F

Let $f(a, b)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}$

Let $g(a)$ be the value of $\mathrm{T}^a\mathrm{F}\square$

Values:

$$
\begin{aligned}
f(0,0) &= -1. \\
f(a,0) &= \{\{a-2 \mid 1\} \mid 0\} &&, \quad a \geq 1. \\
f(a,1) &= \{a-1 \mid 1\} &&, \quad a \geq 0. \\
f(a,b) &= a &&, \quad a \geq 0, b \geq 2.
\end{aligned}
$$

$$
g(a) = 0 \quad , \quad a \geq 0.
$$

ClassA21: □□F

Let $f(a,b,c)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\square\mathrm{T}^c\mathrm{F}$

Let $g(a,b)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}\square$

Let $h(a)$ be the value of $\mathrm{T}^a\mathrm{F}\square\square$

Values:

$$
\begin{aligned}
f(0,0,0) &= -2. \\
f(a,0,0) &= a-1 &&, \quad a \geq 1. \\
f(0,1,0) &= \tfrac{-1}{2}. \\
f(a,1,0) &= a* &&, \quad a \geq 1. \\
f(a,b,0) &= \{\{2a+b-2 \mid a+1\} \mid a\} &&, \quad a \geq 0, \quad b \geq 2. \\
f(0,0,1) &= -1. \\
f(a,0,1) &= a &&, \quad a \geq 1. \\
f(a,b,1) &= \{2a+b-1 \mid a+1\} &&, \quad a \geq 0, \quad b \geq 1. \\
f(a,b,c) &= 2a+b &&, \quad a \geq 0, \quad b \geq 0, \quad c \geq 2.
\end{aligned}
$$

$$
\begin{aligned}
g(0,0) &= -1. \\
g(a,0) &= a - \tfrac{1}{2} &&, \quad a \geq 1. \\
g(a,b) &= a &&, \quad a \geq 0, \quad b \geq 1.
\end{aligned}
$$

$$
h(a) = a \quad , \quad a \geq 0.
$$

ClassA31: □□□F

Let $f(a,b,c,d)$ be the value of T^a□T^b□T^c□T^{d}F.

Let $g(a,b,c)$ be the value of T^a□T^b□T^{c}F□.

Let $h(a,b)$ be the value of T^a□T^{b}F□□.

Let $i(a)$ be the value of T^{a}F□□□.

Values:

$f(0,0,0,0)$	$=$	$-3.$	
$f(a,0,0,0)$	$=$	$(2a-2)*$	, $a \geq 1.$
$f(a,b,0,0)$	$=$	$2a+b-1$	, $a \geq 0,\ \ b \geq 1.$
$f(a,0,1,0)$	$=$	$2a-1$	, $a \geq 0.$
$f(a,b,1,0)$	$=$	$(2a+b)*$	, $a \geq 0,\ \ b \geq 1.$
$f(a,b,c,0)$	$=$	$\{\{3a+2b+c-2 \mid 2a+b+1\} \mid 2a+b\}$	, $a \geq 0,\ \ b \geq 0,\ \ c \geq 2.$
$f(0,0,0,1)$	$=$	$-2.$	
$f(a,0,0,1)$	$=$	$(2a-1)*$	, $a \geq 1.$
$f(a,b,0,1)$	$=$	$2a+b$	, $a \geq 0,\ \ b \geq 1.$
$f(a,b,c,1)$	$=$	$\{3a+2b+c-1 \mid 2a+b+1\}$	, $a \geq 0,\ \ b \geq 0,\ \ c \geq 1.$
$f(a,b,c,d)$	$=$	$3a+2b+c$	, $a \geq 0,\ \ b \geq 0,\ \ c \geq 0,\ \ d \geq 2.$
$g(a,0,0)$	$=$	$2a-2$	, $a \geq 0.$
$g(a,b,0)$	$=$	$(2a+b-1)*.$	
$g(a,b,c)$	$=$	$2a+b$	, $a \geq 0,\ \ b \geq 0,\ \ c \geq 1.$
$h(a,b)$	$=$	$2a+b-1$	, $a \geq 0,\ \ b \geq 0.$
$i(a)$	$=$	$2a$	, $a \geq 0.$

ClassA41: □□□□F

Let $f(a,b,c,d,e)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\square\mathrm{T}^c\square\mathrm{T}^d\square\mathrm{T}^e\mathrm{F}$.

Let $g(a,b,c,d)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\square\mathrm{T}^c\square\mathrm{T}^d\mathrm{F}\square$.

Let $h(a,b,c)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\square\mathrm{T}^c\mathrm{F}\square\square$.

Let $i(a,b)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}\square\square\square$.

Let $j(a)$ be the value of $\mathrm{T}^a\mathrm{F}\square\square\square\square$.

Values:

$$f(0,0,0,0,0) = -4.$$
$$f(a,0,0,0,0) = 3a-3 \quad , \quad a \geq 1.$$
$$f(0,1,0,0,0) = -\tfrac{1}{2}.$$
$$f(a,1,0,0,0) = 3a-\tfrac{1}{4} \quad , \quad a \geq 1.$$
$$f(a,b,0,0,0) = (3a+2b-2)* \quad , \quad a \geq 0, \ b \geq 2.$$
$$f(0,0,1,0,0) = -1.$$
$$f(a,0,1,0,0) = 3a-\tfrac{1}{2} \quad , \quad a \geq 1.$$
$$f(a,b,c,0,0) = 3a+2b+c-1 \quad , \quad a \geq 0, \ b \geq 1, \ c=1 \ \text{or} \ a \geq 0, \ b \geq 0, \ c \geq 2.$$
$$f(0,0,0,1,0) = -2.$$
$$f(a,b,0,1,0) = 3a+2b-1 \quad , \quad a \geq 1, \ b=0 \ \text{or} \ a \geq 0, \ b \geq 1.$$
$$f(0,0,1,1,0) = \tfrac{1}{2}.$$
$$f(a,0,1,1,0) = 3a+\tfrac{3}{4} \quad , \quad a \geq 1.$$
$$f(a,b,c,1,0) = (3a+2b+c)* \quad , \quad a \geq 0, \ b \geq 1, \ c=1 \ \text{or} \ a \geq 0, \ b \geq 0, \ c \geq 2.$$
$$f(0,0,0,2,0) = \{* \mid 0\}.$$
$$f(a,0,0,2,0) = \{\{4a \mid 3a+\tfrac{1}{2}\} \mid 3a\} \quad , \quad a \geq 1.$$
$$f(a,b,c,d,0) = \{\{4a+3b+2c+d-2 \mid 3a+2b+c+1\} \mid 3a+2b+c\}$$
$$, a \geq 0, \ b \geq 1, \ c=0, \ d=2$$
$$\text{or } a \geq 0, \ b \geq 0, \ c \geq 1, \ d=2$$
$$\text{or } a \geq 0, \ b \geq 0, \ c \geq 0, \ d \geq 3.$$

$$
\begin{array}{rcll}
f(0,0,0,0,1) & = & -3. & \\
f(a,0,0,0,1) & = & 3a-2 & ,\quad a \geq 1. \\
f(0,1,0,0,1) & = & \frac{1}{2}. & \\
f(a,1,0,0,1) & = & 3a+\frac{3}{4} & ,\quad a \geq 1. \\
f(a,b,0,0,1) & = & (3a+2b-1)* & ,\quad a \geq 0,\quad b \geq 2. \\
f(0,0,1,0,1) & = & 0. & \\
f(a,0,1,0,1) & = & 3a+\frac{1}{2} & ,\quad a \geq 1. \\
f(a,b,c,0,1) & = & 3a+2b+c & ,\quad a \geq 0,\quad b \geq 1,\quad c=1 \text{ or } a \geq 0,\quad b \geq 0,\quad c \geq 2. \\
f(0,0,0,1,1) & = & *. & \\
f(a,0,0,1,1) & = & \{4a \mid 3a+\frac{1}{2}\} & ,\quad a \geq 1.
\end{array}
$$

$$
\begin{array}{l}
f(a,b,c,d,1) = \{4a+3b+2c+d-1 \mid 3a+2b+c+1\} \\
\qquad ,a \geq 0,\quad b \geq 1,\quad c=0,\quad d=1 \quad \text{or} \\
\qquad a \geq 0,\quad b \geq 0,\quad c \geq 1,\quad d=1 \quad \text{or} \\
\qquad a \geq 0,\quad b \geq 0,\quad c \geq 0,\quad d \geq 2.
\end{array}
$$

$$
f(a,b,c,d,e) = 4a+3b+2c+d \quad ,\quad a \geq 0,\quad b \geq 0,\quad c \geq 0,\quad d \geq 0,\quad e \geq 2.
$$

$$
\begin{array}{rcll}
g(0,0,0,0) & = & -3. & \\
g(a,0,0,0) & = & 3a-\frac{5}{2} & ,\quad a \geq 1. \\
g(a,b,0,0) & = & 3a+2b-2 & ,\quad a \geq 0,\quad b \geq 1. \\
g(0,0,1,0) & = & -\frac{1}{2}. & \\
g(a,0,1,0) & = & 3a-\frac{1}{4} & ,\quad a \geq 1. \\
g(a,b,c,0) & = & (3a+2b+c-1)* & ,\quad a \geq 0,\quad b \geq 1,\quad c=1 \\
 & & & \text{or}\quad a \geq 0,\quad b \geq 0,\quad c \geq 2. \\
g(0,0,0,1) & = & -1. & \\
g(a,0,0,1) & = & 3a-\frac{1}{2} & ,\quad a \geq 1.
\end{array}
$$

$$
\begin{array}{lcll}
g(a,b,c,d) & = & 3a+2b+c & ,\ a \geq 0,\ \ b \geq 1,\ \ c = 0,\ \ d = 1 \\
 & & & \text{or}\ \ a \geq 0,\ \ b \geq 0,\ \ c \geq 1,\ \ d = 1 \\
 & & & \text{or}\ \ a \geq 0,\ \ b \geq 0,\ \ c \geq 0,\ \ d \geq 2.
\end{array}
$$

$$
\begin{array}{lcll}
h(a,0,0) & = & 3a-2 & ,\ a \geq 0. \\
h(a,b,0) & = & 3a+2b-\frac{3}{2} & ,\ a \geq 0,\ \ b \geq 1. \\
h(a,b,c) & = & 3a+2b+c-1 & ,\ a \geq 0,\ \ b \geq 0,\ \ c \geq 1.
\end{array}
$$

$$
i(a,b) = 3a+2b-1 \quad ,\ a \geq 0,\ \ b \geq 0.
$$

$$
j(a) = 3a \quad ,\ a \geq 0.
$$

ClassA51: □□□□□F

Let $f(a,b,c,d,e,l)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\square\mathrm{T}^c\square\mathrm{T}^d\square\mathrm{T}^e\square\mathrm{T}^l\mathrm{F}$.

Let $g(a,b,c,d,e)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\square\mathrm{T}^c\square\mathrm{T}^d\square\mathrm{T}^e\mathrm{F}\square$.

Let $h(a,b,c,d)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\square\mathrm{T}^c\square\mathrm{T}^d\mathrm{F}\square\square$.

Let $i(a,b,c)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\square\mathrm{T}^c\mathrm{F}\square\square\square$.

Let $j(a,b)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}\square\square\square\square$.

Let $k(a)$ be the value of $\mathrm{T}^a\mathrm{F}\square\square\square\square\square$.

Values:

$$
\begin{array}{lcll}
f(0,0,0,0,0,0) & = & -5. & \\
f(a,0,0,0,0,0) & = & (4a-4)* & ,\ a \geq 1. \\
f(a,b,0,0,0,0) & = & 4a+3b-3 & ,\ a \geq 0,\ \ b \geq 1. \\
f(a,b,1,0,0,0) & = & 4a+3b-1 & ,\ a \geq 0,\ \ b \geq 0. \\
f(a,b,c,0,0,0) & = & (4a+3b+2c-2)* & ,\ a \geq 0,\ \ b \geq 0,\ \ c \geq 2. \\
f(a,0,0,1,0,0) & = & 4a-2 & ,\ a \geq 0. \\
f(a,b,0,1,0,0) & = & (4a+3b-1)* & ,\ a \geq 0,\ \ b \geq 1.
\end{array}
$$

$$
\begin{aligned}
f(a,b,c,d,0,0) &= 4a+3b+2c+d-1 &&, \ a\geq 0,\ \ b\geq 0,\ \ c\geq 1,\ \ d=1\\
&&& \text{or}\ \ a\geq 0,\ \ b\geq 0,\ \ c\geq 0,\ \ d\geq 2.\\
f(0,0,0,0,1,0) &= -3.\\
f(a,0,0,0,1,0) &= (4a-2)* &&, \ a\geq 1.\\
f(a,b,c,0,1,0) &= 4a+3b+2c-1 &&, \ a\geq 0,\ \ b\geq 1,\ \ c=0\\
&&& \text{or}\ \ a\geq 0,\ \ b\geq 0,\ \ c\geq 1,.\\
f(a,b,0,1,1,0) &= 4a+3b &&, \ a\geq 0,\ \ b\geq 0.\\
f(a,b,c,d,1,0) &= (4a+3b+2c+d)* &&, \ a\geq 0,\ \ b\geq 0,\ \ c\geq 1,\ \ d=1\\
&&& \text{or}\ \ a\geq 0,\ \ b\geq 0,\ \ c\geq 0,\ \ d\geq 2.\\
f(a,0,0,0,2,0) &= 4a-1 &&, \ a\geq 0.\\
f(a,b,0,0,2,0) &= (4a+3b)* &&, \ a\geq 0,\ \ b\geq 1.
\end{aligned}
$$

$$
\begin{aligned}
f(a,b,c,d,e,0) = \{\{5a+4b+3c+2d+e-2 \mid 4a+3b+2c+d+1\} \mid 4a+3b+2c+d\}\\
, \ \ a\geq 0,\ \ b\geq 0,\ \ c\geq 1,\ \ d=0,\ \ e=2\\
\text{or}\ \ a\geq 0,\ \ b\geq 0,\ \ c\geq 0,\ \ d\geq 1,\ \ e=2\\
\text{or}\ \ a\geq 0,\ \ b\geq 0,\ \ c\geq 0,\ \ d\geq 0,\ \ e\geq 3.
\end{aligned}
$$

$$
f(a,b,c,d,0,1) = f(a,b,c,d,0,0)+1 \quad , \ a\geq 0,\ \ b\geq 0,\ \ c\geq 0,\ \ d\geq 0.
$$

$$
\begin{aligned}
f(a,0,0,0,1,1) &= \{5a \mid 4a-1\} &&, \ a\geq 0.\\
f(a,b,0,0,1,1) &= \{5a+4b \mid (4a+3b)*\} &&, \ a\geq 0,\ \ b\geq 1.
\end{aligned}
$$

$$
\begin{aligned}
f(a,b,c,d,e,1) = \{5a+4b+3c+2d+e-1 \mid 4a+3b+2c+d+1\}\ ,\\
a\geq 0,\ \ b\geq 0,\ \ c\geq 1,\ \ d=0,\ \ e=1\\
\text{or}\ \ a\geq 0,\ \ b\geq 0,\ \ c\geq 0,\ \ d\geq 1,\ \ e=1\\
\text{or}\ \ a\geq 0,\ \ b\geq 0,\ \ c\geq 0,\ \ d\geq 0,\ \ e\geq 2\ .
\end{aligned}
$$

$$
f(a,b,c,d,e,l) = 5a+4b+3c+2d+e,\ \ a\geq 0,\ \ b\geq 0,\ \ c\geq 0,\ \ d\geq 0, e\geq 0,\ \ l\geq 2.
$$

$$
\begin{array}{lcll}
g(a,0,0,0,0) & = & 4a-4 & ,\ a \ge 0. \\
g(a,b,0,0,0) & = & (4a+3b-3)* & ,\ a \ge 0,\ \ b \ge 1. \\
g(a,b,c,0,0) & = & 4a+3b+2c-2 & ,\ a \ge 0,\ \ b \ge 0,\ \ c \ge 1. \\
g(a,b,0,1,0) & = & 4a+3b-1 & ,\ a \ge 0,\ \ b \ge 0. \\
g(a,b,c,d,0) & = & (4a+3b+2c+d-1)* & ,\ a \ge 0,\ \ b \ge 0,\ \ c \ge 1,\ \ d = 1 \\
 & & & \text{or}\ \ a \ge 0,\ \ b \ge 0,\ \ c \ge 0,\ \ d \ge 2.
\end{array}
$$

$$
\begin{array}{lcll}
g(a,0,0,0,1) & = & 4a-2 & ,\ a \ge 0. \\
g(a,b,0,0,1) & = & (4a+3b-1)* & ,\ a \ge 0,\ \ b \ge 1. \\
g(a,b,c,d,e) & = & 4a+3b+2c+d & ,\ a \ge 0,\ \ b \ge 0,\ \ c \ge 1,\ \ d = 0,\ \ e = 1 \\
 & & & \text{or}\ \ a \ge 0,\ \ b \ge 0,\ \ c \ge 0,\ \ d \ge 1,\ \ e = 1 \\
 & & & \text{or}\ \ a \ge 0,\ \ b \ge 0,\ \ c \ge 0,\ \ d \ge 0,\ \ e \ge 2.
\end{array}
$$

$$
\begin{array}{lcll}
h(a,b,0,0) & = & 4a+3b-3 & ,\ a \ge 0,\ \ b \ge 0. \\
h(a,b,c,0) & = & (4a+3b+2c-2)* & ,\ a \ge 0,\ \ b \ge 0,\ \ c \ge 1. \\
h(a,b,c,d) & = & 4a+3b+2c+d-1 & ,\ a \ge 0,\ \ b \ge 0,\ \ c \ge 0,\ \ d \ge 1.
\end{array}
$$

$$
i(a,b,c) = 4a+3b+2c-2 \quad ,\ a \ge 0,\ \ b \ge 0,\ \ c \ge 0.
$$

$$
j(a,b) = 4a+3b-1 \quad ,\ a \ge 0,\ \ b \ge 0.
$$

$$
k(a) = 4a \quad ,\ a \ge 0.
$$

3.3 Result of class with two frogs.

ClassA12: □FF

Let $f(a,b,c)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}\mathrm{T}^c\mathrm{F}$

Let $g(a,b,c)$ be the value of $\mathrm{T}^a\mathrm{F}\mathrm{T}^b\square\mathrm{T}^c\mathrm{F}$

Let $h(a, b)$ be the value of $\mathrm{T}^a\mathrm{F}\mathrm{T}^b\mathrm{F}\square$

Values:

$$
\begin{array}{lcll}
f(0,0,0) & = & -2. & \\
f(0,0,c) & = & -1 & ,\ c \geq 1. \\
f(1,0,0) & = & \{0 \mid -\frac{1}{2}\}. & \\
f(a,0,c) & = & \{\{a-2 \mid \{0 \mid c\}\} \mid 0\} & ,\ a = 1,\ c \geq 1 \text{ or } a \geq 2,\ c \geq 0. \\
f(a,1,c) & = & \{a-1 \mid \{0 \mid c\}\} & ,\ a \geq 0,\ c \geq 0. \\
f(a,b,c) & = & a & ,\ a \geq 0,\ b \geq 2,\ c \geq 0.
\end{array}
$$

$$
\begin{array}{lcll}
g(0,0,0) & = & -1. & \\
g(1,0,0) & = & -\frac{1}{2}. & \\
g(a,0,0) & = & \{\{\{a-3 \mid \frac{1}{2}\} \mid 0\} \mid 0\} & ,\ a \geq 2. \\
g(a,b,0) & = & \{\{b-2 \mid 1\} \mid 0\} & ,\ a \geq 0,\ b \geq 1. \\
g(a,b,1) & = & \{b-1 \mid 1\} & ,\ a \geq 0,\ b \geq 0. \\
g(a,b,c) & = & b & ,\ a \geq 0,\ b \geq 0,\ c \geq 2.
\end{array}
$$

$$
\begin{array}{lcll}
h(a,b) & = & 0 & ,\ a \geq 0,\ b \geq 0.
\end{array}
$$

ClassA22: $\square\square\mathrm{FF}$

Let $f(a, b, c, d)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\square\mathrm{T}^c\mathrm{F}\mathrm{T}^d\mathrm{F}$.

Let $g(a, b, c, d)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}\mathrm{T}^c\square\mathrm{T}^d\mathrm{F}$.

Let $h(a, b, c, d)$ be the value of $\mathrm{T}^a\mathrm{F}\mathrm{T}^b\square\mathrm{T}^c\square\mathrm{T}^d\mathrm{F}$.

Let $i(a, b, c)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}\mathrm{T}^c\mathrm{F}\square$.

Let $j(a, b, c)$ be the value of $\mathrm{T}^a\mathrm{F}\mathrm{T}^b\square\mathrm{T}^c\mathrm{F}\square$.

Let $k(a, b)$ be the value of $\mathrm{T}^a\mathrm{F}\mathrm{T}^b\mathrm{F}\square\square$.

Note: In f, g, h, we omit the case where d ≥ 2 since it will reduce to ClassA21.

Values:

For d=0

$$
\begin{aligned}
f(0,0,0,0) &= -4. \\
f(1,0,0,0) &= -1. \\
f(2,0,0,0) &= *. \\
f(a,0,0,0) &= \{a - \tfrac{5}{2} \mid 0\} && , \quad a \geq 3. \\
f(0,1,0,0) &= \tfrac{-3}{2}. \\
f(1,1,0,0) &= 0. \\
f(2,1,0,0) &= \{1 \mid \{\tfrac{1}{2} \mid 0\}\}. \\
f(a,1,0,0) &= a - \tfrac{3}{2} && , \quad a \geq 3. \\
f(0,2,0,0) &= \{\{0 \mid *\} \mid \tfrac{-1}{4}\}. \\
f(1,2,0,0) &= \tfrac{1}{2} * . \\
f(a,2,0,0) &= a - 1 && , \quad a \geq 2. \\
f(a,b,0,0) &= a* && , \quad a \geq 0, \quad b \geq 3.
\end{aligned}
$$

$$
\begin{aligned}
f(0,0,1,0) &= -2. \\
f(1,0,1,0) &= *. \\
f(2,0,1,0) &= 1. \\
f(3,0,1,0) &= 1. \\
f(a,0,1,0) &= \{(a-2)* \mid 1*\} && , \quad a \geq 4. \\
f(0,1,1,0) &= \{0 \mid *\}. \\
f(1,1,1,0) &= \{2 \mid (\tfrac{1}{2})*\}. \\
f(a,1,1,0) &= \{2a \mid \{(a-1)* \mid 1*\}\} && , \quad a \geq 2. \\
f(a,b,1,0) &= \{2a + b - 1 \mid a*\} && , \quad a \geq 0, \quad b \geq 2.
\end{aligned}
$$

For d=1

$$
\begin{aligned}
f(0,0,0,1) &= -3. \\
f(a,0,0,1) &= \{a - 2 \mid 1\} && , \quad a \geq 1.
\end{aligned}
$$

$$
\begin{aligned}
f(0,1,0,1) &= -\tfrac{1}{2}. \\
f(a,1,0,1) &= \{a-\tfrac{1}{2} \mid \{a-1 \mid 1\}\} & , \quad & a \ge 1. \\
f(a,2,0,1) &= \{\{2a \mid \{a \mid \{a \mid 2\}\}\} \mid a\} & , \quad & a \ge 0. \\
f(a,b,0,1) &= \{\{2a+b-2 \mid a+\tfrac{1}{2}\} \mid a\} & , \quad & a \ge 0, \quad b \ge 3. \\
f(0,0,1,1) &= -1. \\
f(a,0,1,1) &= a-\tfrac{1}{2} & , \quad & a \ge 1 \\
f(a,1,1,1) &= \{2a \mid \{a \mid \{a \mid 2\}\}\} & , \quad & a \ge 0. \\
f(a,b,1,1) &= \{2a+b-1 \mid a+\tfrac{1}{2}\} & , \quad & a \ge 0, \quad b \ge 2. \\
f(a,b,c,d) &= 2a+b & , \quad & a \ge 0, \quad b \ge 0, \quad c \ge 2, \quad d \ge 0.
\end{aligned}
$$

For d=0

First $a = 0, \quad b = 0.$

$$
\begin{aligned}
g(0,0,0,0) &= -3. \\
g(0,0,1,0) &= -\tfrac{3}{2}. \\
g(0,0,c,0) &= \{\{c-3 \mid 0\} \mid -1\} & , \quad & c \ge 2 \\
g(0,0,0,1) &= -2. \\
g(0,0,c,1) &= \{c-2 \mid 0\} & , \quad & c \ge 1 \\
g(0,0,c,d) &= c-1 & , \quad & c \ge 0, \quad d \ge 2.
\end{aligned}
$$

Second $c = 0, \quad d = 0.$

$$
\begin{aligned}
g(1,0,0,0) &= (-1)\,*\,. \\
g(a,0,0,0) &= 0 & , \quad & a \ge 2. \\
g(0,1,0,0) &= -1. \\
g(1,1,0,0) &= *. \\
g(a,1,0,0) &= \{a-\tfrac{3}{2} \mid 0\} & , \quad & a \ge 2.
\end{aligned}
$$

$$
\begin{aligned}
g(0,2,0,0) &= -\tfrac{1}{4}. \\
g(a,2,0,0) &= a-\tfrac{1}{2} &&, \quad a \geq 1. \\
g(a,3,0,0) &= \{\{\{2a \mid \{a \mid \{a \mid 2\}\}\} \mid a\} \mid a\} &&, \quad a \geq 0. \\
g(a,b,0,0) &= \{\{\{2a+b-3 \mid a+\tfrac{1}{2}\} \mid a\} \mid a\} &&, \quad a \geq 0, \quad b \geq 4.
\end{aligned}
$$

Third $b=0, \quad d=0.$

$$
\begin{aligned}
g(1,0,1,0) &= 0. \\
g(2,0,1,0) &= \{1 \mid \{\tfrac{1}{2} \mid 0\}\}. \\
g(a,0,1,0) &= 1* &&, \quad a \geq 3. \\
g(a,0,c,0) &= \{\{a+c-\tfrac{5}{2} \mid \{\{a-1 \mid 2\} \mid 1\}\} \mid \{\{a-2 \mid 1\} \mid 0\}\} &&, \quad a \geq 1, \quad c \geq 2.
\end{aligned}
$$

Fourth $b=1, \quad d=0.$

$$
\begin{aligned}
g(0,1,1,0) &= *. \\
g(1,1,1,0) &= \tfrac{1}{2}*. \\
g(a,1,1,0) &= \{(a-1)* \mid 1*\} &&, \quad a \geq 2. \\
g(a,1,c,0) &= \{\{a+c-2 \mid \{a \mid 2\}\} \mid \{a-1 \mid 1\}, 2\} &&, \quad (a,c)=(0,1) \text{ or } a \geq 0, \quad c \geq 2.
\end{aligned}
$$

Last $b \geq 2, d=0.$

$$
g(a,b,c,0) = \{\{a+c-2 \mid a+1\} \mid a\} \quad , \quad a \geq 0, \quad b \geq 2, \quad c \geq 1.
$$

Now for d=1

First $b=0, \quad d=1.$

$$
\begin{aligned}
g(1,0,0,1) &= *. \\
g(a,0,0,1) &= 1 &&, \quad a \geq 2. \\
g(a,0,c,1) &= \{\{a+c-\tfrac{3}{2} \mid \{a-1 \mid 2\} \mid 1\}\} &&, \quad a \geq 1, \quad c \geq 1.
\end{aligned}
$$

Second $b = 1, \quad d = 1.$

$$g(a,1,0,1) = \{a-1 \mid 1\} \quad , \quad a \geq 0.$$
$$g(a,1,c,1) = \{a+c-1 \mid \{a \mid 2\}\} \quad , \quad a \geq 0, \quad c \geq 1.$$

Last $b \geq 2, \quad d = 1.$

$$g(a,b,c,1) = \{a+c-1 \mid a+1\} \quad , \quad a \geq 0, \quad b \geq 2, \quad c \geq 0.$$

$$h(0,0,0,0) = -2.$$
$$h(1,0,0,0) = -1.$$
$$h(2,0,0,0) = *.$$
$$h(3,0,0,0) = \{\tfrac{1}{2} \mid 0\}.$$
$$h(a,0,0,0) = \{1* \mid 0\} \quad , \quad a \geq 4.$$
$$h(a,b,0,0) = b-1 \quad , \quad a \geq 0, \quad b \geq 1.$$
$$h(0,0,1,0) = -\tfrac{1}{2}.$$
$$h(1,0,1,0) = *.$$
$$h(a,0,1,0) = \{\{\{a-2 \mid 2\} \mid 1\} \mid 0\} \quad , \quad a \geq 2.$$
$$h(a,b,1,0) = b* \quad , \quad a \geq 0, \quad b \geq 1.$$
$$h(a,b,c,0) = \{\{a+2b+c-2 \mid b+1\} \mid b\} \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2.$$

$$h(0,0,0,1) = -1.$$
$$h(1,0,0,1) = 0.$$
$$h(a,0,0,1) = \{\{a-2 \mid 2\} \mid 1\} \quad , \quad a \geq 2.$$
$$h(a,b,0,1) = b \quad , \quad a \geq 0, \quad b \geq 1.$$
$$h(a,b,c,1) = \{a+2b+c-1 \mid b+1\} \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 1.$$

$$
\begin{array}{lcll}
i(0,0,0) & = & -2. & \\
i(1,0,0) & = & \{0 \mid \frac{-1}{2}\}. & \\
i(a,0,0) & = & \{\{a-2 \mid \frac{1}{2}\} \mid \{\{\{a-3 \mid 1\} \mid 0\} \mid 0\}\} & ,\ a \geq 2. \\
i(0,0,c) & = & -1 & ,\ c \geq 1. \\
i(a,0,c) & = & \{\{a-2 \mid 1\} \mid 0\} & ,\ a \geq 1,\ \ c \geq 1. \\
i(a,1,0) & = & \{a-1 \mid \frac{1}{2}\} & ,\ a \geq 0. \\
i(a,1,c) & = & \{a-1 \mid 1\} & ,\ a \geq 0,\ \ c \geq 1. \\
i(a,b,c) & = & a & ,\ a \geq 0,\ \ b \geq 2,\ \ c \geq 0.
\end{array}
$$

$$
\begin{array}{lcll}
j(0,0,0) & = & -1. & \\
j(1,0,0) & = & \frac{-1}{2}. & \\
j(a,0,0) & = & \{\{\{a-3 \mid 1\} \mid 0\} \mid 0\} & ,\ a \geq 2. \\
j(a,b,0) & = & b-\frac{1}{2} & ,\ a \geq 0,\ \ b \geq 1. \\
j(a,b,c) & = & b & ,\ a \geq 0,\ \ b \geq 0,\ \ c \geq 1.
\end{array}
$$

$$
k(a,b) = b \quad ,\ a \geq 0,\ \ b \geq 0.
$$

ClassA32: □□□FF

Let $f(a,b,c,d,e)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\square\mathrm{T}^c\square\mathrm{T}^d\mathrm{F}\mathrm{T}^e\mathrm{F}$.

Let $g(a,b,c,d,e)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\square\mathrm{T}^c\mathrm{F}\mathrm{T}^d\square\mathrm{T}^e\mathrm{F}$.

Let $h(a,b,c,d,e)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}\mathrm{T}^c\square\mathrm{T}^d\square\mathrm{T}^e\mathrm{F}$.

Let $i(a,b,c,d,e)$ be the value of $\mathrm{T}^a\mathrm{F}\mathrm{T}^b\square\mathrm{T}^c\square\mathrm{T}^d\square\mathrm{T}^e\mathrm{F}$.

Let $j(a,b,c,d)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\square\mathrm{T}^c\mathrm{F}\mathrm{T}^d\mathrm{F}\square$.

Let $k(a,b,c,d)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}\mathrm{T}^c\square\mathrm{T}^d\mathrm{F}\square$.

Let $l(a,b,c,d)$ be the value of $\mathrm{T}^a\mathrm{F}\mathrm{T}^b\square\mathrm{T}^c\square\mathrm{T}^d\mathrm{F}\square$.

Let $m(a,b,c)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}\mathrm{T}^c\mathrm{F}\square\square$.

Let $n(a,b,c)$ be the value of $\mathrm{T}^a\mathrm{F}\mathrm{T}^b\square\mathrm{T}^c\mathrm{F}\square\square$.

Let $o(a,b)$ be the value of $\mathrm{T}^a\mathrm{F}\mathrm{T}^b\mathrm{F}\square\square\square$.

Note: In f, g, h, i we omit the case when e ≥ 2 since they will reduce to ClassA31.

Values:

First for $e = 0$

$$
\begin{aligned}
f(0,0,0,0,0) &= -6. \\
f(1,0,0,0,0) &= (-2)*. \\
f(a,0,0,0,0) &= (a-2)* && , \quad a \geq 2. \\
f(0,1,0,0,0) &= -2. \\
f(a,1,0,0,0) &= a-1 && , \quad a \geq 1. \\
f(a,2,0,0,0) &= \{2a-1 \mid a\} && , \quad a \geq 0. \\
f(a,b,0,0,0) &= \{2a+b-\tfrac{5}{2} \mid a\} && , \quad a \geq 0, \quad b \geq 3. \\
f(0,0,1,0,0) &= -3. \\
f(a,0,1,0,0) &= \{2a-3 \mid a-1\} && , \quad a \geq 1. \\
f(a,1,1,0,0) &= 2a-1 && , \quad a \geq 0. \\
f(a,b,1,0,0) &= 2a+b-\tfrac{3}{2} && , \quad a \geq 0, \quad b \geq 2. \\
f(0,0,2,0,0) &= (-1)*. \\
f(1,0,2,0,0) &= 1*. \\
f(a,0,2,0,0) &= 2a-\tfrac{3}{2} && , \quad a \geq 2. \\
f(0,1,2,0,0) &= \tfrac{1}{2}*. \\
f(a,1,2,0,0) &= 2a && , \quad a \geq 1. \\
f(a,b,2,0,0) &= 2a+b-1 && , \quad a \geq 0, \quad b \geq 2. \\
f(a,b,c,0,0) &= (2a+b)* && , \quad a \geq 0, \quad b \geq 0, \quad c \geq 3.
\end{aligned}
$$

$$
\begin{aligned}
f(0,0,0,1,0) &= -4. \\
f(1,0,0,1,0) &= -1. \\
f(a,0,0,1,0) &= a-1 && , \quad a \geq 2. \\
f(0,1,0,1,0) &= (-1)*. \\
f(1,1,0,1,0) &= 1*.
\end{aligned}
$$

$$f(a,1,0,1,0) = \{\{(2a-2)* \mid a+\tfrac{1}{2}\} \mid a\} \quad , \quad a \geq 2.$$

$$f(0,2,0,1,0) = \tfrac{1}{2}*.$$

$$f(a,b,0,1,0) = \{(2a+b-2)* \mid (a+1)*\} \quad , \quad a \geq 1, \quad b = 2 \text{ or } a \geq 0, \quad b \geq 3.$$

$$f(0,0,1,1,0) = \{0 \mid (-1)*\}.$$

$$f(1,0,1,1,0) = \{3 \mid 1*\}.$$

$$f(a,0,1,1,0) = \{3a \mid \{\{(2a-2)* \mid a+\tfrac{1}{2}\} \mid a\}\} \quad , \quad a \geq 2.$$

$$f(0,1,1,1,0) = \{2 \mid \tfrac{1}{2}\}.$$

$$f(a,b,1,1,0) = \{3a+2b \mid \{(2a+b-1)* \mid (a+1)*\}\} \quad , \quad a \geq 1, \quad b = 1 \text{ or } a \geq 0, \quad b \geq 2.$$

$$f(a,b,c,1,0) = \{3a+2b+c-1 \mid (2a+b)*\} \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2.$$

$$f(a,b,c,d,0) = 3a+2b+c \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 0, \quad d \geq 2.$$

Second for $e = 1$

$$f(0,0,0,0,1) = -5.$$

$$f(1,0,0,0,1) = (-1)*.$$

$$f(2,0,0,0,1) = 1*.$$

$$f(a,0,0,0,1) = \{\{(2a-4)* \mid a-\tfrac{1}{2}\} \mid a-1\} \quad , \quad a \geq 3.$$

$$f(0,1,0,0,1) = -1.$$

$$f(1,1,0,0,1) = 1.$$

$$f(a,1,0,0,1) = \{(2a-2)* \mid a+\tfrac{1}{2}\} \quad , \quad a \geq 2.$$

$$f(a,b,0,0,1) = \{2a+b-2 \mid a+1\} \quad , \quad a \geq 0, \quad b \geq 2.$$

$$f(0,0,1,0,1) = -\tfrac{3}{2}.$$

$$f(1,0,1,0,1) = \tfrac{1}{2}.$$

$$f(a,0,1,0,1) = (2a-2)* \quad , \quad a \geq 2.$$

$$f(a,1,1,0,1) = 2a \quad , \quad a \geq 0.$$

$$f(a,b,1,0,1) = \{\{2a+b-1 \mid \{2a+b-1 \mid a+2\}\} \mid \{2a+b-1 \mid a+1\}\} \quad , \quad a \geq 0, \quad b \geq 2.$$

$$f(a,0,2,0,1) = (2a)* \quad , \quad a \geq 0.$$

$$f(a,b,2,0,1) = \{\{3a+2b \mid \{2a+b \mid \{2a+b \mid a+2\}\}\} \mid 2a+b\} \quad , \quad a \geq 0, \quad b \geq 1.$$

$$f(a,b,c,0,1) = \{\{3a+2b+c-2 \mid 2a+b+\tfrac{1}{2}\} \mid 2a+b\} \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 3.$$

$$
\begin{aligned}
f(a,0,0,1,1) &= 2a-2 && , \quad a \geq 0. \\
f(a,1,0,1,1) &= (2a)* && , \quad a \geq 0. \\
f(a,b,0,1,1) &= \{2a+b-1 \mid \\
&\quad \{(2a+b-1)*, \{2a+b-1 \mid \{2a+b-1 \mid a+2\}\} \mid \\
&\quad \{2a+b-1 \mid a+1\}, \{\{2a+b-1 \mid a+2\} \mid a+1\}\}\} && , \quad a \geq 0, \quad b \geq 2. \\
f(0,0,1,1,1) &= \{0 \mid *\}. \\
f(a,0,1,1,1) &= \{3a \mid \{2a \mid a+1\}\} && , \quad a \geq 1. \\
f(a,b,1,1,1) &= \{3a+2b \mid \{2a+b \mid \{2a+b \mid a+2\}\}\} && , \quad a \geq 0, \quad b \geq 1. \\
f(a,b,c,1,1) &= \{3a+2b+c-1 \mid 2a+b+\tfrac{1}{2}\} && , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2. \\
f(a,b,c,d,1) &= 3a+2b+c && , \quad a \geq 0, \quad b \geq 0, \quad c \geq 0, \quad d \geq 2.
\end{aligned}
$$

For $e = 0$, **I)** $a = 0, \quad b = 0, \quad c = 0$

$$
\begin{aligned}
g(0,0,0,0,0) &= -5. \\
g(0,0,0,1,0) &= -3. \\
g(0,0,0,d,0) &= \{\{d-4 \mid -1\} \mid -2\} && , \quad d \geq 2.
\end{aligned}
$$

For $e = 0$, **II)** $d = 0$

$$
\begin{aligned}
g(1,0,0,0,0) &= -2. \\
g(a,0,0,0,0) &= a-2 && , \quad a \geq 2. \\
g(0,1,0,0,0) &= (-2)*. \\
g(a,1,0,0,0) &= (a-1)* && , \quad a \geq 1. \\
g(0,2,0,0,0) &= (-\tfrac{1}{2})*. \\
g(a,2,0,0,0) &= a && , \quad a \geq 1. \\
g(a,b,0,0,0) &= a && , \quad a \geq 0, \quad b \geq 3. \\
g(0,0,1,0,0) &= -2.
\end{aligned}
$$

$$g(a,0,1,0,0) = a-1 \quad , \quad a \geq 1.$$

$$g(a,1,1,0,0) = \{2a-1 \mid a\} \quad , \quad a \geq 0.$$

$$g(a,b,1,0,0) = \{2a+b-\tfrac{3}{2} \mid a\} \quad , \quad a \geq 0, \quad b \geq 2.$$

$$g(a,0,2,0,0) = 2a-1 \quad , \quad a \geq 0.$$

$$g(a,b,2,0,0) = 2a+b-\tfrac{1}{2} \quad , \quad a \geq 0, \quad b \geq 1.$$

$$g(a,0,3,0,0) = \{(2a)* \mid 2a\} \quad , \quad a \geq 0.$$

$$g(a,b,3,0,0) = \{\{\{3a+2b \mid \{2a+b \mid \{2a+b \mid a+2\}\}\} \mid 2a+b\} \mid 2a+b\} \quad , \quad a \geq 0, \quad b \geq 1.$$

$$g(a,b,c,0,0) = \{\{\{3a+2b+c-3 \mid 2a+b+\tfrac{1}{2}\} \mid 2a+b\} \mid 2a+b\} \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 4.$$

For $e=0$, **III)** $c=0$

$$g(1,0,0,1,0) = -\tfrac{1}{2}.$$

$$g(a,0,0,1,0) = (a-1)* \quad , \quad a \geq 2.$$

$$g(a,0,0,d,0) = \{\{2a+d-4 \mid a\} \mid a-1\} \quad , \quad a \geq 1, \quad d \geq 2.$$

$$g(0,1,0,1,0) = -1.$$

$$g(0,1,0,d,0) = \{\{(d-2)* \mid \tfrac{1}{2}\} \mid -\tfrac{1}{2}\} \quad , \quad d \geq 2.$$

$$g(a,1,0,1,0) = a \quad , \quad a \geq 1.$$

$$g(a,1,0,d,0) = \{\{(2a+d-2)* \mid (a+1)*\} \mid a*\} \quad , \quad a \geq 1, \quad d \geq 2.$$

$$g(0,2,0,1,0) = \tfrac{1}{2}.$$

$$g(a,2,0,1,0) = (a+1)* \quad , \quad a \geq 1.$$

$$g(a,b,0,1,0) = (a+1)* \quad , \quad a \geq 0, \quad b \geq 3.$$

$$\begin{aligned} g(a,b,0,2,0) = \ & \{\{(2a+b-1)*, \{2a+b-1 \mid \{2a+b-1 \mid a+2\}\} \mid \\ & \{\{2a+b-1 \mid a+2\} \mid a+1\}, \{2a+b-1 \mid a+1\}\} \mid \\ & \{\{2a+b-2 \mid a+1\} \mid a\}\} \quad , \quad a \geq 0, \quad b \geq 2. \end{aligned}$$

$$\begin{aligned} g(a,b,0,d,0) = \ & \{\{(2a+b+d-3)* \mid \{\{2a+b-1 \mid a+2\} \mid a+1\}\} \mid \\ & \{\{2a+b-2 \mid a+1\} \mid a\}\} \quad , \quad a \geq 0, \quad b \geq 2, \quad d \geq 3. \end{aligned}$$

For $e = 0$, **IV)** $c = 1$

$$
\begin{aligned}
g(0,0,1,1,0) &= (-1)*. \\
g(1,0,1,1,0) &= 1*. \\
g(a,0,1,1,0) &= \{\{(2a-2)* \mid a+\tfrac{1}{2}\} \mid a\} && , \quad a \geq 2. \\
g(0,1,1,1,0) &= \tfrac{1}{2}*. \\
g(a,b,1,1,0) &= \{(2a+b-1)* \mid (a+1)*\} && , \quad a \geq 1, \quad b = 1 \text{ or } a \geq 0, \quad b \geq 2. \\
g(0,0,1,d,0) &= \{\{d-2 \mid 0\} \mid -1\} && , \quad d \geq 2. \\
g(a,0,1,d,0) &= \{\{2a+d-2 \mid a+1\} \mid a\} && , \quad a \geq 1, \quad d \geq 2. \\
g(a,b,1,d,0) &= \{\{2a+b+d-2 \mid \{2a+b \mid a+2\}\} \mid \\
&\quad \{2a+b-1 \mid a+1\}, a+2\} && , \quad a \geq 0, \quad b \geq 1, \quad d \geq 2.
\end{aligned}
$$

For $e = 0$, **V)** $c \geq 1$

$$
\begin{aligned}
g(a,b,c,1,0) &= (2a+b)* && , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2. \\
g(a,b,c,d,0) &= \{\{2a+b+d-2 \mid 2a+b+1\} \mid 2a+b\} && , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2, \quad d \geq 2.
\end{aligned}
$$

For $e = 1$, **I)** $a = 0, \quad b = 0, \quad c = 0$

$$
\begin{aligned}
g(0,0,0,0,1) &= -4. \\
g(0,0,0,d,1) &= \{d-3 \mid -1\} && , \quad d \geq 1.
\end{aligned}
$$

For $e = 1$, **II)** $d = 0$

$$
\begin{aligned}
g(1,0,0,0,1) &= -1. \\
g(a,0,0,0,1) &= a-1 && , \quad a \geq 2. \\
g(0,1,0,0,1) &= (-1)*. \\
g(1,1,0,0,1) &= 1*. \\
g(a,1,0,0,1) &= \{\{(2a-2)* \mid a+\tfrac{1}{2}\} \mid a\} && , \quad a \geq 2. \\
g(0,2,0,0,1) &= (\tfrac{1}{2})*. \\
g(a,2,0,0,1) &= a+1 && , \quad a \geq 1. \\
g(a,b,0,0,1) &= a+1 && , \quad a \geq 0, \quad b \geq 3.
\end{aligned}
$$

$$
\begin{aligned}
g(0,0,1,0,1) &= -1. \\
g(1,0,1,0,1) &= 1. \\
g(a,0,1,0,1) &= \{(2a-2)* \mid a+\tfrac{1}{2}\} \quad , \quad a \geq 2. \\
g(a,b,1,0,1) &= \{2a+b-1 \mid a+1\} \quad , \quad a \geq 0, \quad b \geq 1. \\
g(a,b,c,0,1) &= 2a+b \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2.
\end{aligned}
$$

For $e = 1$, **III)** $c = 0$

$$
\begin{aligned}
g(a,0,0,d,1) &= \{2a+d-3 \mid a\} \quad , \quad a \geq 1, \quad d \geq 1. \\
g(0,1,0,d,1) &= \{(d-1)* \mid \tfrac{1}{2}\} \quad , \quad d \geq 1. \\
g(a,1,0,d,1) &= \{(2a+d-1)* \mid (a+1)*\} \quad , \quad a \geq 1, \quad d \geq 1. \\
g(a,b,0,1,1) &= \{(2a+b-1)*, \{2a+b-1 \mid \{2a+b-1 \mid a+2\}\} \mid \\
&\quad\ \{2a+b-1 \mid a+1\}, \{\{2a+b-1 \mid a+2\} \mid a+1\}\} \quad , \quad a \geq 0, \quad b \geq 2. \\
g(a,b,0,d,1) &= \{(2a+b+d-2)* \mid \{\{2a+b-1 \mid a+2\} \mid a+1\}\} \quad , \quad a \geq 0, \quad b \geq 2, \quad d \geq 2.
\end{aligned}
$$

For $e = 1$, **IV)** $c = 1$

$$
\begin{aligned}
g(0,0,1,d,1) &= \{d-1 \mid 0\} \quad , \quad d \geq 1. \\
g(a,0,1,d,1) &= \{2a+d-1 \mid a+1\} \quad , \quad a \geq 1, \quad d \geq 1. \\
g(a,b,1,d,1) &= \{2a+b+d-1 \mid \{2a+b \mid a+2\}\} \quad , \quad a \geq 0, \quad b \geq 1, \quad d \geq 1.
\end{aligned}
$$

For $e = 1$, **V)** $c \geq 2$

$$
g(a,b,c,d,1) = \{2a+b+d-1 \mid 2a+b+1\} \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2, \quad d \geq 1.
$$

For $e = 0$, **I)** $a = 0, b = 0$

$$
\begin{aligned}
h(0,0,0,0,0) &= -4. \\
h(0,0,c,0,0) &= c-2 \quad , \quad c \geq 1. \\
h(0,0,0,1,0) &= -2. \\
h(0,0,c,1,0) &= (c-1)* \quad , \quad c \geq 1. \\
h(0,0,c,d,0) &= \{\{2c+d-3 \mid c\} \mid c-1\} \quad , \quad c \geq 0, \quad d \geq 2.
\end{aligned}
$$

For $e = 0$, **II)** $c = 0$, $d = 0$

$$
\begin{aligned}
h(1,0,0,0,0) &= -1. \\
h(a,0,0,0,0) &= \{a-2 \mid \{a-2 \mid 3\}\} &&, \quad a \geq 2. \\
h(0,1,0,0,0) &= -2. \\
h(a,1,0,0,0) &= a-1 &&, \quad a \geq 1. \\
h(0,2,0,0,0) &= -\tfrac{1}{2}. \\
h(a,2,0,0,0) &= a* &&, \quad a \geq 1. \\
h(0,3,0,0,0) &= \{\tfrac{1}{2} \mid 0\}. \\
h(a,3,0,0,0) &= \{(a+1)* \mid a\} &&, \quad a \geq 1. \\
h(a,b,0,0,0) &= \{(a+1)* \mid a\} &&, \quad a \geq 0, \quad b \geq 4.
\end{aligned}
$$

For $e = 0$, **III)** $b = 0$

$$
\begin{aligned}
h(a,0,c,0,0) &= a+c-\tfrac{3}{2} &&, \quad a \geq 1, \quad c \geq 1. \\
h(1,0,0,1,0) &= *. \\
h(2,0,0,1,0) &= \tfrac{3}{2}*. \\
h(3,0,0,1,0) &= \tfrac{5}{2}*. \\
h(4,0,0,1,0) &= \tfrac{27}{8}.
\end{aligned}
$$

$$
\begin{aligned}
h(a,0,0,1,0) &= \{a-1 \mid \{a-1 \mid 5\}\} &&, \quad a \geq 5. \\
h(a,0,c,1,0) &= (a+c-\tfrac{1}{2})* &&, \quad a \geq 1, \quad c \geq 1. \\
h(a,0,0,2,0) &= \{\{2a-1 \mid a+\tfrac{1}{2}\} \mid a-\tfrac{1}{2}\} &&, \quad a = 1,2,3,4,5. \\
h(6,0,0,2,0) &= \{\{11 \mid 6*\} \mid \tfrac{11}{2}\}. \\
h(a,0,0,2,0) &= \{a \mid a-\tfrac{1}{2}\} &&, \quad a \geq 7. \\
h(a,0,c,d,0) &= \{\{2a+2c+d-3 \mid a+c+\tfrac{1}{2}\} \mid a+c-\tfrac{1}{2}\} &&, \quad a \geq 1, \quad c \geq 1, \quad d = 2 \\
& &&\text{or} \quad a \geq 1, \quad c \geq 0, \quad d \geq 3.
\end{aligned}
$$

For $e = 0$, **IV)** $d = 0,1$, $b \geq 1$

$$
\begin{aligned}
h(a,b,c,0,0) &= a+c-1 &&, \quad a \geq 0, \quad b \geq 1, \quad c \geq 1. \\
h(0,1,0,1,0) &= -\tfrac{1}{2}. \\
h(a,1,0,1,0) &= a* &&, \quad a \geq 1.
\end{aligned}
$$

$$
\begin{aligned}
h(0,2,0,1,0) &= \{\tfrac{1}{2} \mid 0\}. \\
h(a,2,0,1,0) &= \{(a+1)* \mid a\} && , \quad a \geq 1. \\
h(a,b,0,1,0) &= \{\{\{2a+b-2*, \{2a+b-2 \mid \\
&\quad \{2a+b-2 \mid a+2\}\} \mid \{2a+b-2 \mid a+1\}, \\
&\quad \{\{2a+b-2 \mid a+2\} \mid a+1\}\} \mid a+1\} \mid a\} \\
& && , \quad a \geq 0, \quad b \geq 3. \\
h(a,b,c,1,0) &= (a+c)* && , \quad a \geq 0, \quad b \geq 1, \quad c \geq 1.
\end{aligned}
$$

For $e = 0$, **V)** $d \geq 2, \quad b \geq 1$

$$
h(a,b,c,d,0) = \{\{2a+b+2c+d-3 \mid a+c+1\} \mid a+c\} \quad , \quad a \geq 0, \quad b \geq 1, \quad c \geq 0,, \quad d \geq 2.
$$

For $e = 1$, **I)** $a = 0, b = 0$

$$
\begin{aligned}
h(0,0,0,0,1) &= -3. \\
h(0,0,c,0,1) &= c-1 && , \quad c \geq 1. \\
h(0,0,c,d,1) &= \{2c+d-2 \mid c\} && , \quad c \geq 0, \quad d \geq 1.
\end{aligned}
$$

For $e = 1$, **II)** $c = 0, d = 0$

$$
\begin{aligned}
h(1,0,0,0,1) &= 0. \\
h(a,0,0,0,1) &= \{a-1 \mid \{a-1 \mid 4\}\} && , \quad a \geq 2. \\
h(0,1,0,0,1) &= -1. \\
h(a,1,0,0,1) &= a && , \quad a \geq 1. \\
h(0,2,0,0,1) &= \tfrac{1}{2}. \\
h(a,2,0,0,1) &= (a+1)* && , \quad a \geq 1. \\
h(a,b,0,0,1) &= \{\{2a+b-2*, \{2a+b-2 \mid \{2a+b-2 \mid a+2\}\} \\
&\quad \mid \{\{2a+b-2 \mid a+2\} \mid a+1\}, \\
&\quad \{2a+b-2 \mid a+1\}\} \mid a+1\} \\
&\quad , \quad a \geq 0, \quad b \geq 3.
\end{aligned}
$$

For $e=1$, **III)** $b=0$

$$h(a,0,c,0,1) = a+c-\tfrac{1}{2} \quad , \quad a\geq 1, \quad c\geq 1.$$
$$h(a,0,0,1,1) = \{2a-1 \mid a+\tfrac{1}{2}\} \quad , \quad 1\leq a\leq 4.$$
$$h(a,0,0,1,1) = \{2a-1 \mid \{a \mid 6\}\} \quad , \quad a\geq 5.$$
$$h(a,0,c,1,1) = \{2a+2c-1 \mid a+c+\tfrac{1}{2}\} \quad , \quad a\geq 1, \quad c\geq 1.$$
$$h(a,0,c,d,1) = \{2a+2c+d-2 \mid a+c+\tfrac{1}{2}\} \quad , \quad a\geq 1, \quad c\geq 0, \quad d\geq 2$$

For $e=1$, **IV)** $d=0$, $b\geq 1$

$$h(a,b,c,0,1) = a+c \quad , \quad a\geq 0, \quad b\geq 1, \quad c\geq 1.$$

For $e=1$, **V)** $d\geq 1$, $b\geq 1$

$$h(a,b,c,d,1) = \{2a+b+2c+d-2 \mid a+c+1\} \quad , \quad a\geq 0, \quad b\geq 1, \quad c\geq 0,, \quad d\geq 1.$$

First for $d=0$, $e=0$

$$i(0,0,0,0,0) = -3.$$
$$i(1,0,0,0,0) = (-1)*.$$
$$i(a,0,0,0,0) = a-1 \quad , \quad a\geq 2.$$
$$i(a,1,0,0,0) = a* \quad , \quad a\geq 0.$$
$$i(a,b,0,0,0) = \{a+2b-2 \mid a+2b-2\} \quad , \quad a=0,1,2, \quad b\geq 2.$$
$$i(3,b,0,0,0) = (2b+\tfrac{1}{2})* \quad , \quad b\geq 2.$$
$$i(a,b,0,0,0) = a+2b-3 \quad , \quad a\geq 4, \quad b\geq 2.$$

$$i(a,b,c,0,0) = a+2b+c-1 \quad , \quad a=0,1,2, \quad b\geq 0, \quad c\geq 1.$$
$$i(3,0,1,0,0) = 3.$$
$$i(3,b,1,0,0) = 2b+\tfrac{5}{2} \quad , \quad b\geq 1.$$
$$i(3,b,c,0,0) = 2b+c+2 \quad , \quad b\geq 0, \quad c\geq 2.$$
$$i(a,0,1,0,0) = a \quad , \quad a\geq 4.$$
$$i(a,b,1,0,0) = (a+2b-1)* \quad , \quad a\geq 4, \quad b\geq 1.$$
$$i(a,b,c,0,0) = a+2b+c-1 \quad , \quad a\geq 0, \quad b\geq 0, \quad c\geq 2.$$

Second for $d \geq 1, \quad e = 0$

$$\begin{aligned}
i(0,0,0,1,0) &= -1. \\
i(1,0,0,1,0) &= 0. \\
i(a,0,0,1,0) &= a* && , \quad a \geq 2. \\
i(a,b,0,1,0) &= a + 2b - 1 && , \quad a \geq 0, \quad b \geq 1.
\end{aligned}$$

$$\begin{aligned}
i(a,b,1,1,0) &= (a + 2b + 1)* && , \quad a = 0,1,2, \quad b \geq 0. \\
i(3,0,1,1,0) &= 4*. \\
i(3,b,1,1,0) &= 2b + \tfrac{15}{4} && , \quad b \geq 1. \\
i(a,0,1,1,0) &= (a+1)* && , \quad a \geq 4. \\
i(a,b,1,1,0) &= a + 2b && , \quad a \geq 4, \quad b \geq 1. \\
i(a,b,c,1,0) &= (a + 2b + c)* && , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2.
\end{aligned}$$

$$\begin{aligned}
i(a,b,c,2,0) &= \{\{2a + 3b + 2c \mid a + 2b + c + 1\} \mid a + 2b + c\} && , \quad a = 0,1,2, \quad b \geq 0, \quad c \geq 0. \\
i(3,0,0,2,0) &= \{\{6 \mid 4\} \mid 3\}. \\
i(3,b,0,2,0) &= \{\{3b + 6 \mid 2b + \tfrac{7}{2}\} \mid 2b + 3\} && , \quad b \geq 1. \\
i(3,b,c,2,0) &= \{\{3b + 2c + 6 \mid 2b + c + 4\} \mid 2b + c + 3\} && , \quad b \geq 0, \quad c \geq 1. \\
i(a,0,0,2,0) &= \{\{2a \mid a + 1\} \mid a\} && , \quad a \geq 4. \\
i(a,b,0,2,0) &= (a + 2b)* && , \quad a \geq 4, \quad b \geq 1. \\
i(a,b,c,2,0) &= \{\{2a + 3b + 2c \mid a + 2b + c + 1\} \mid a + 2b + c\} && , \quad a \geq 4, \quad b \geq 0, \quad c \geq 1. \\
i(a,b,c,d,0) &= \{\{2a + 3b + 2c + d - 2 \mid a + 2b + c + 1\} \mid a + 2b + c\} && , \quad a \geq 0, \quad b \geq 0, \quad c \geq 0, \quad d \geq 3.
\end{aligned}$$

Third for $d = 0, \quad e = 1$

$$\begin{aligned}
i(0,0,0,0,1) &= -2. \\
i(1,0,0,0,1) &= *. \\
i(a,0,0,0,1) &= a && , \quad a \geq 2. \\
i(a,1,0,0,1) &= (a+1)* && , \quad a \geq 0.
\end{aligned}$$

$$i(a,b,0,0,1) = \{a+2b-1 \mid a+2b-1\} \quad , \quad a=0,1,2, \quad b \geq 2.$$
$$i(3,b,0,0,1) = (2b+\tfrac{3}{2})* \quad , \quad b \geq 2.$$
$$i(a,b,0,0,1) = a+2b-2 \quad , \quad a \geq 4, \quad b \geq 2.$$

$$i(a,b,c,0,1) = a+2b+c \quad , \quad a=0,1,2, \quad b \geq 0, \quad c \geq 1.$$
$$i(3,0,1,0,1) = 4.$$
$$i(3,b,1,0,1) = 2b+\tfrac{7}{2} \quad , \quad b \geq 1.$$
$$i(3,b,c,0,1) = 2b+c+3 \quad , \quad b \geq 0, \quad c \geq 2.$$
$$i(a,0,1,0,1) = a+1 \quad , \quad a \geq 4.$$
$$i(a,b,1,0,1) = (a+2b)* \quad , \quad a \geq 4, \quad b \geq 1.$$
$$i(a,b,c,0,1) = a+2b+c \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2.$$

Note: $i(a,b,c,0,1) = i(a,b,c,0,0)+1, \quad a \geq 0, \quad b \geq 0, \quad c \geq 0.$

Fourth for $d \geq 1, \quad e=1$

$$i(a,0,0,1,1) = \{2a \mid a+1\} \quad , \quad a \geq 0.$$
$$i(a,b,0,1,1) = \{2a+3b \mid \{a+2b \mid 2b+4\}\} \quad , \quad a \geq 0, \quad b \geq 1.$$
$$i(a,b,c,1,1) = \{2a+3b+2c \mid a+2b+c+1\} \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 1.$$
$$i(a,b,c,d,1) = \{2a+3b+2c+d-1 \mid a+2b+c+1\} \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 0, \quad d \geq 2.$$

First for $c=0, \quad d=0$

$$j(0,0,0,0) = -4.$$
$$j(1,0,0,0) = -1.$$
$$j(a,0,0,0) = \{a-2 \mid \{a-2 \mid 3\}\} \quad , \quad a \geq 2.$$
$$j(0,1,0,0) = (-1)*.$$
$$j(a,1,0,0) = \{a-\tfrac{1}{2} \mid \{a-1 \mid \{a-1 \mid 3\}\}\} \quad , \quad a \geq 1.$$
$$j(0,2,0,0) = \{\tfrac{1}{4} \mid -\tfrac{1}{4}\}.$$
$$j(a,2,0,0) = \{\{2a \mid a+\tfrac{1}{2}\} \mid \{a* \mid a\}\} \quad , \quad a \geq 1.$$
$$j(a,b,0,0) = \{\{2a+b-2 \mid a+\tfrac{1}{2}\} \mid \{\{\{2a+b-3 \mid a+1\} \mid a\} \mid a\}\} \quad , \quad a \geq 0, \quad b \geq 3.$$

Second for $c = 0, \quad d \geq 1$

$$
\begin{aligned}
j(0,0,0,d) &= -2 &&, \quad d \geq 1. \\
j(a,0,0,d) &= a-1 &&, \quad a \geq 1, \quad d \geq 1. \\
j(0,1,0,d) &= -\tfrac{1}{2} &&, \quad d \geq 1. \\
j(a,1,0,d) &= a* &&, \quad a \geq 1, \quad d \geq 1. \\
j(a,b,0,d) &= \{\{2a+b-2 \mid a+1\} \mid a\} &&, \quad a \geq 0, \quad b \geq 2, \quad d \geq 1.
\end{aligned}
$$

Third for $c = 1, \quad d = 0$

$$
\begin{aligned}
j(0,0,1,0) &= -1. \\
j(a,0,1,0) &= a - \tfrac{1}{2} &&, \quad a \geq 1. \\
j(a,b,1,0) &= \{2a+b-1 \mid a+\tfrac{1}{2}\} &&, \quad a \geq 0, \quad b \geq 1.
\end{aligned}
$$

Fourth for $c = 1, \quad d \geq 1$

$$
\begin{aligned}
j(0,0,1,d) &= -1 &&, \quad d \geq 1. \\
j(a,0,1,d) &= a &&, \quad a \geq 1, \quad d \geq 1. \\
j(a,b,1,d) &= \{2a+b-1 \mid a+1\} &&, \quad a \geq 0, \quad b \geq 1, \quad d \geq 1.
\end{aligned}
$$

Fifth for $c \geq 2$

$$
j(a,b,c,d) = 2a+b \quad , \quad a \geq 0, \quad b \geq 3, \quad c \geq 2, \quad d \geq 0.
$$

First for $b = 0, \quad d = 0$

$$
\begin{aligned}
k(0,0,0,0) &= -3. \\
k(1,0,0,0) &= (-1)*. \\
k(a,0,0,0) &= \{a-2 \mid 3\} &&, \quad a \geq 2. \\
k(0,0,c,0) &= (c-2)* &&, \quad c \geq 1. \\
k(a,0,c,0) &= a+c-1 &&, \quad a \geq 1, \quad c \geq 1.
\end{aligned}
$$

Second for $b \geq 1, \quad d = 0$

$$\begin{aligned}
k(0,1,0,0) &= -1. \\
k(a,1,0,0) &= \{a-1 \mid \{a-1 \mid 3\}\} &&, \quad a \geq 1. \\
k(0,2,0,0) &= -\tfrac{1}{4}. \\
k(a,2,0,0) &= \{a* \mid a\} &&, \quad a \geq 1. \\
k(a,b,0,0) &= \{\{\{2a+b-3 \mid a+1\} \mid a\} \mid a\} &&, \quad a \geq 0, \quad b \geq 3. \\
k(a,b,c,0) &= a+c-\tfrac{1}{2} &&, \quad a \geq 0, \quad b \geq 1, \quad c \geq 1.
\end{aligned}$$

Third for $b = 0, \quad d \geq 1$

$$\begin{aligned}
k(0,0,c,d) &= c-1 &&, \quad c \geq 0, \quad d \geq 1. \\
k(a,0,0,1) &= a-\tfrac{1}{2} &&, \quad a = 1,2,3,4. \\
k(a,0,0,1) &= \{a-1 \mid 5\} &&, \quad a \geq 5. \\
k(a,0,c,d) &= a+c-\tfrac{1}{2} &&, \quad a \geq 1, \quad c \geq 1, \quad d = 1 \\
& && \text{or} \quad a \geq 1, \quad c \geq 0, \quad d \geq 2.
\end{aligned}$$

Fourth for $b \geq 1, \quad d \geq 1$

$$k(a,b,c,d) = a+c \qquad , \quad a \geq 0, \quad b \geq 1, \quad c \geq 0, \quad d \geq 1.$$

$$\begin{aligned}
l(0,0,0,0) &= -2. \\
l(1,0,0,0) &= -1. \\
l(a,0,0,0) &= (a-1)* &&, \quad a \geq 2. \\
l(a,1,0,0) &= \{a* \mid 3\} &&, \quad a \geq 0. \\
l(a,b,0,0) &= \{a+2b-3 \mid 2b+1\} &&, \quad a \geq 0, \quad b \geq 2. \\
l(a,0,1,0) &= a* &&, \quad a \geq 0.
\end{aligned}$$

$$\begin{aligned}
l(a,b,1,0) &= (a+2b)* &&, \quad a = 0,1,2, \quad b \geq 1. \\
l(3,b,1,0) &= (2b+\tfrac{5}{2})* &&, \quad b \geq 1. \\
l(a,b,1,0) &= a+2b-1 &&, \quad a \geq 4, \quad b \geq 1.
\end{aligned}$$

$$
\begin{aligned}
l(a,b,c,0) &= (a+2b+c-1)* && , \; a \geq 0, \; b \geq 0, \; c \geq 2. \\
l(a,0,0,1) &= a && , \; a \geq 0. \\
l(a,b,0,1) &= \{a+2b-1 \mid 2b+3\} && , \; a \geq 0, \; b \geq 1. \\
l(a,b,c,d) &= a+2b+c && , \; a \geq 0, \; b \geq 0, \; c \geq 1, \; d = 1 \\
& && \text{or } \; a \geq 0, \; b \geq 0, \; c \geq 0, \; d \geq 2.
\end{aligned}
$$

First for $b = 0, \;\; a = 0$

$$m(0,0,c) = c-2 \quad , \quad c \geq 0.$$

Second for $b = 0, \;\; c = 0 \;\; , (a \geq 1)$

$$
\begin{aligned}
m(1,0,0) &= \{0 \mid -\tfrac{1}{2}\}. \\
m(2,0,0) &= 1*. \\
m(3,0,0) &= 2*. \\
m(4,0,0) &= \tfrac{11}{4}. \\
m(a,0,0) &= 3 && , \; a \geq 5.
\end{aligned}
$$

$$
\begin{aligned}
m(a,0,c) &= (a+c-1)* && , \; a \geq 1, \; c \geq 1. \\
m(a,1,0) &= \{a-1 \mid 3\} && , \; a \geq 0. \\
m(a,1,c) &= a+c && , \; a \geq 0, \; c \geq 1. \\
m(a,b,c) &= a+c && , \; a \geq 0, \; b \geq 2, \; c \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
n(0,0,0) &= -1. \\
n(1,0,0) &= -\tfrac{1}{2}. \\
n(a,0,0) &= \{(a-1)* \mid 0\} && , \; a \geq 2. \\
n(a,1,0) &= \{\{a* \mid 3\} \mid 2\} && , \; a \geq 0. \\
n(a,b,0) &= \{\{a+2b-3 \mid 2b+1\} \mid 2b\} && , \; a \geq 0, \; b \geq 2.
\end{aligned}
$$

$$
\begin{array}{lcll}
n(a,0,1) & = & \{a* \mid 3\} & , \quad a \geq 0. \\
n(a,b,1) & = & \{a+2b-1 \mid 2b+3\} & , \quad a \geq 0, \quad b \geq 1. \\
n(a,b,c) & = & a+2b+c-1 & , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2. \\
 & & & \\
o(a,b) & = & 2b & , \quad a \geq 0, \quad b \geq 0.
\end{array}
$$

3.4 Positions with one □

In this section we classify all the positions that have one □. The lemmas below give a recurrence to the positions. We can compute the values of the positions in polynomial time. We omit the proofs here.

Notation

$O(x) = \{0 \mid x\}$.

$O^a(x) = O(...(O(O(x))))$ a times.

$\widetilde{\mathrm{L}}$ = any combination of T and F that has F as its rightmost entry. For example TTFTF.

$\widetilde{\mathrm{R}}$ = any combination of T and F that has T as its left most entry. For example TFFTF.

Lemma 3.4.1. *Death Leaps Principle(DLP): The position with one empty square for which the only possible move for both sides is a jump has value 0. For example* TFTTF□TFTFF.

Lemma 3.4.2. $\widetilde{\mathrm{L}}\mathrm{T}\Box\mathrm{F}\widetilde{\mathrm{R}} = *$.

Lemma 3.4.3. $\widetilde{\mathrm{L}}\mathrm{T}^a\Box\mathrm{F}^b\widetilde{\mathrm{R}} = *, \quad a \geq 2, b \geq 2$.

Lemma 3.4.4. $\widetilde{\mathrm{L}}\mathrm{T}^a\Box(\mathrm{TF})^b\mathrm{TF}^c\widetilde{\mathrm{R}} = \{a-1 \mid (O^b(\widetilde{\mathrm{L}}\mathrm{T}^a\mathrm{F}(\mathrm{TF})^b\mathrm{T}\Box\mathrm{F}^{c-1}\widetilde{\mathrm{R}})\}, \quad a \geq 1, b \geq 0, c \geq 2$.

$\widetilde{\mathrm{L}}\mathrm{T}^a\Box\mathrm{F}(\mathrm{TF})^b\mathrm{TF}^c\widetilde{\mathrm{R}} = \{\{a-2 \mid O^{b+1}(\widetilde{\mathrm{L}}\mathrm{T}^{a-1}\mathrm{F}(\mathrm{TF})^{b+1}\mathrm{T}\Box\mathrm{F}^{c-1}\widetilde{\mathrm{R}})\} \mid\mid 0\}, \quad a \geq 2, b \geq 0, c \geq 2$.

Example 1: $\mathrm{T}^a\Box\mathrm{F}(\mathrm{TF})^b\mathrm{TF}^2 = \{\{a-2 \mid O^{b+1}(*)\} \mid\mid 0\},\ \ a \geq 1, b \geq 0.$

Example 2: $\mathrm{T}^3\Box\mathrm{F}(\mathrm{TF})^b\mathrm{TF}^c = \{\{1 \mid O^{b+1}(\mathrm{T}^2\mathrm{F}(\mathrm{TF})^{b+1}\mathrm{T}\Box\mathrm{F}^{c-1})\} \mid\mid 0\},\ \ b \geq 0, c \geq 2.$

Example 3: $\mathrm{T}^a\Box\mathrm{F}(\mathrm{TF})^b\mathrm{TF}^4 = \{\{a-2 \mid O^{b+1}(\mathrm{T}^{a-1}\mathrm{F}(\mathrm{TF})^{b+1}\mathrm{T}\Box\mathrm{F}^3)\} \mid\mid 0\},\ \ a \geq 3, b \geq 0.$

Note) We get the implicit value of example 2 from example 1 and implicit value of example 3 from example 2. We will get the value recursively this way.

Lemma 3.4.5. $\widetilde{\mathrm{L}}\mathrm{T}^a\Box(\mathrm{TF})^b = \{a-1 \mid (\frac{1}{2})^{b-1}\},\ \ a \geq 1, b \geq 1.$

Lemma 3.4.6. $\widetilde{\mathrm{L}}\mathrm{T}^a\Box\mathrm{F}(\mathrm{TF})^b = \{\{a-2 \mid (\frac{1}{2})^b\} \mid\mid 0\},\ \ a \geq 2, b \geq 0.$

Lemma 3.4.7. $\widetilde{\mathrm{L}}\mathrm{T}^a\Box\mathrm{F}(\mathrm{TF})^b\mathrm{TF}^c\widetilde{\mathrm{R}} = \mathrm{T}^a\Box\mathrm{F}(\mathrm{TF})^b\mathrm{TF}^c,\ \ b \geq 0$

when (c is even and $a \geq c-1$) or (a is odd and $c \geq a-1$).

Note:

1) When a is even ,$a \geq 2$ and c is odd ,$c \geq 3$, The recursive is going to bounce back and forth between positions.
We will refer to $\widetilde{\mathrm{R}}$ if $a > c$. We will refer to $\widetilde{\mathrm{L}}$ if $c > a$. Then the positions will start over again.

2) When a is even , c is even and $a < c-1$ then we will refer to $\widetilde{\mathrm{L}}$.
When a is odd , c is odd and $c < a-1$ then we will refer to $\widetilde{\mathrm{R}}$.

3.5 Result of class with three frogs.

ClassA13: □FFF

Let $f(a,b,c,d)$ be the value of $\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}\mathrm{T}^c\mathrm{F}\mathrm{T}^d\mathrm{F}$.

Let $g(a,b,c,d)$ be the value of $\mathrm{T}^a\mathrm{F}\mathrm{T}^b\square\mathrm{T}^c\mathrm{F}\mathrm{T}^d\mathrm{F}$.

Let $h(a,b,c,d)$ be the value of $\mathrm{T}^a\mathrm{F}\mathrm{T}^b\mathrm{F}\mathrm{T}^c\square\mathrm{T}^d\mathrm{F}$.

Let $i(a,b,c)$ be the value of $\mathrm{T}^a\mathrm{F}\mathrm{T}^b\mathrm{F}\mathrm{T}^c\mathrm{F}\square$.

Note:

1) In f we omit the case where $c \geq 2$ or $d \geq 2$ since it will reduce to the results in ClassA11 and ClassA12 respectively.

2) In g, h we omit the case where $d \geq 2$ since it will reduce to the results of ClassA12.

$$f(0,0,0,0) = -3.$$
$$f(1,0,0,0) = \{0 \mid (-1)*\}.$$
$$f(a,0,0,0) = * \quad , \quad a \geq 2.$$
$$f(0,1,0,0) = 0.$$
$$f(1,1,0,0) = \{0 \mid \{0 \mid -\tfrac{1}{4}\}\}.$$
$$f(a,1,0,0) = \{a-1 \mid \{0 \mid \{\{\{\{a-3 \mid \tfrac{1}{4}\} \mid 0\} \mid 0\} \mid 0\}\}\} \quad , \quad a \geq 2.$$
$$f(a,b,0,0) = a \quad , \quad a \geq 0, \quad b \geq 2.$$
$$f(0,0,1,0) = -1.$$
$$f(a,0,1,0) = \{\{a-2 \mid \{0 \mid *\}\} \mid 0\} \quad , \quad a \geq 1.$$
$$f(a,1,1,0) = \{a-1 \mid \{0 \mid *\}\} \quad , \quad a \geq 0.$$
$$f(a,b,1,0) = a \quad , \quad a \geq 0, \quad b \geq 2.$$

$$f(0,0,0,1) = -2.$$
$$f(1,0,0,1) = \{0 \mid -\tfrac{1}{2}\}.$$
$$f(a,0,0,1) = * \quad , \quad a \geq 2.$$

$$f(a,1,0,1) = \{a-1 \mid *\} \quad , \quad a \geq 0.$$

$$f(a,b,0,1) = a \quad , \quad a \geq 0, \quad b \geq 2.$$

$$f(0,0,1,1) = -1.$$

$$f(a,0,1,1) = \{\{a-2 \mid \tfrac{1}{4}\} \mid 0\} \quad , \quad a \geq 1.$$

$$f(a,1,1,1) = \{a-1 \mid \tfrac{1}{4}\} \quad , \quad a \geq 0.$$

$$f(a,b,1,1) = a \quad , \quad a \geq 0, \quad b \geq 2.$$

$$g(0,0,0,0) = -2.$$

$$g(1,0,0,0) = (-1)*.$$

$$g(a,0,0,0) = \{\{\{a-3 \mid \{0 \mid *\}\} \mid 0\} \mid -1\} \quad , \quad a \geq 2.$$

$$g(0,1,0,0) = \{0 \mid -\tfrac{1}{2}\}.$$

$$g(1,1,0,0) = \{0 \mid -\tfrac{1}{4}\}.$$

$$g(a,1,0,0) = \{0 \mid \{\{\{\{a-3 \mid \tfrac{1}{4}\} \mid 0\} \mid 0\} \mid 0\}\} \quad , \quad a \geq 2.$$

$$g(a,b,0,0) = * \quad , \quad a \geq 0, \quad b \geq 2.$$

$$g(a,b,1,0) = \{b-1 \mid *\} \quad , \quad a \geq 0, \quad b \geq 0.$$

$$g(0,0,0,1) = -1.$$

$$g(1,0,0,1) = -\tfrac{1}{2}.$$

$$g(a,0,0,1) = \{\{\{a-3 \mid \tfrac{1}{4}\} \mid 0\} \mid 0\} \quad , \quad a \geq 2.$$

$$g(a,b,0,1) = \{\{b-2 \mid \tfrac{1}{2}\} \mid 0\} \quad , \quad a \geq 0, \quad b \geq 1.$$

$$g(a,b,1,1) = \{b-1 \mid \tfrac{1}{2}\} \quad , \quad a \geq 0, \quad b \geq 0.$$

$$g(a,b,c,d) = b \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 2, \quad d \geq 0.$$

$$h(a,0,0,0) = -1 \quad , \quad a \geq 0.$$

$$h(a,0,c,0) = \{\{c-2 \mid 1\} \mid 0\} \quad , \quad a \geq 0, \quad c \geq 1.$$

$$h(0,1,0,0) = -\tfrac{1}{2}.$$

$$h(1,1,0,0) = -\tfrac{1}{4}.$$

$$h(a,1,0,0) = \{\{\{\{a-3 \mid \tfrac{1}{4}\} \mid 0\} \mid 0\} \mid 0\} \quad , \quad a \geq 2.$$

$$h(a,b,0,0) = \{\{\{b-3 \mid \tfrac{1}{2}\} \mid 0\} \mid 0\} \quad , \quad a \geq 0, \quad b \geq 2.$$

$$h(a,b,c,0) = \{\{c-2 \mid 1\} \mid 0\} \quad , \quad a \geq 0, \quad b \geq 1, \quad c \geq 1.$$

$$
\begin{aligned}
h(a,0,c,1) &= \{c-1 \mid 1\} &&, \quad a \geq 0, \quad c \geq 0. \\
h(a,b,0,1) &= 0 &&, \quad a \geq 0, \quad b \geq 1. \\
h(a,b,1,1) &= \tfrac{1}{2} &&, \quad a \geq 0, \quad b \geq 1. \\
h(a,b,c,1) &= \{c-1 \mid 1\} &&, \quad a \geq 0, \quad b \geq 1, \quad c \geq 2.
\end{aligned}
$$

$$
i(a,b,c) = 0 \quad , \quad a \geq 0, \quad b \geq 0, \quad c \geq 0.
$$

3.6 Result of class B.

ClassB11: □F

Let $f(a,b,c)$ be the value of $\square^a \mathrm{T} \square^b \mathrm{F} \square^c$.

Values:

$$
\begin{aligned}
f(a,b,c) &= \{c-a-2 \mid c-a+2\} &&, \quad a \geq 0, \quad c \geq 0 \text{ and b is even.} \\
f(a,b,c) &= \{\{c-a-3 \mid c-a+1\} \mid \{c-a-1 \mid c-a+3\}\} &&, \quad a \geq 0, \quad c \geq 0 \text{ and b is odd.}
\end{aligned}
$$

Chapter 4

Further Hopping with Toads and Frogs

4.1 Introduction

In this chapter we investigate the patterns of the values of positions of Toads and Frogs which do not follow from the finite state method discussed in the previous two chapters. We prove new values, new conjectures, and outline future work.

4.2 The general classes A and B

Definition:

General class Ai: All positions with (numeric) i number of □ (with symbols on both T and F).

General class Bi: All positions with (numeric) i number of F (with symbols on both T and □).

The general classes A and B are generalizations of the classes A and B discussed in chapter 2.

For the general class, we can not apply the finite state method that we used in chapter 2 since we now have infinitely many positions that come from the combination of the two letters with symbols on them. However we managed to categorize all positions in the general class A1, the class of all positions with exactly on □. It is in fact the only general class that we managed to solve.

Many positions in these classes do not have a nice compact formula; for example in

A2, $\mathrm{T}^a\square\mathrm{TF}\square\mathrm{TF}^b$. On the other hand, many positions have a nice formula. We will prove some of the starting positions like $\mathrm{T}^a\square\square\mathrm{F}^a$, $\mathrm{T}^a\square\square\mathrm{F}^b$ later on in the appendix to this chapter.

Once we detect the patterns of the positions, the proof is quite routine. We now do the proof for each specific position by hand with the help of a computer. We hope to see the computer playing a more active role in assisting with the proofs in the future.

4.3 Table

We present the values of some starting positions in this section. We have a fast program written in Java to calculate the outcome of the sum of two given positions (=,>,<,||). This program does not calculate the value of the sum of two games. It only gives the outcome. It works well with the positions that have a simple value. The author's brother and the author wrote this program originally to check the value of the game of the form $\mathrm{T}^a\square^b\mathrm{F}^a$ for which so far the values of the game are 0 or * except the column b=2 which will be proved to be infinitesimal when $a \geq 4$. The program can be downloaded from the author's websit,e and we present the tables here.

$a\backslash b$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0
2	$*$	$*$	$*$	$*$	0	0	$*$	0	0	0	0	0	$*$	0	0	0	0	0	$*$	0
3	$*$	$\pm\frac{1}{8}$	0	$*$	0	$*$	0	0	0	0	0	0	0	$*$	0	0	0	0	0	0
4	$*$	N	$*$	0	0	0	0	$*$	$*$	0	0	0	0	0	0	0	0	0	0	0
5	$*$	N	$*$	$*$	$*$	0	$*$	0	0	0	$\neq 0$	0								
6	$*$	N	$*$	$*$	$*$	$*$	$*$	$\neq 0$												
7	$*$	N	$*$	$*$	$*$	$\neq 0$	$\neq 0$													
8	$*$	N	$*$	$*$	$*$															
9	$*$	N	$*$	$*$																
10	$*$	N	$*$	$*$																

Figure 4.1: $\mathrm{T}^a\square^b\mathrm{F}^a$

Note 1) For b where $21 \leq b \leq 103$, $\mathrm{T}^2\square^b\mathrm{F}^2 = 0$ except b = 25,31,37,43,49,55,61,67,73,79,85,91,97,103.

Note 2) For b where $21 \leq b \leq 53, \mathrm{T}^3\Box^b\mathrm{F}^3 = 0$ except b = 29.

Note 3) N is an infinitesimal, it is long. We are not writing it out here.

$a\backslash b$	3	4	5	6
1	$2*$	3	$4*$	5
2	$1*$	$2*$	3	$\frac{7}{2}$
3	$\{1 \mid \{\frac{1}{2} \mid 0\}\}$	$1*$	2	$\frac{11}{4}$
4	$1*$	$\{\{2* \mid 1*\} \mid\mid \{\frac{1}{2} \mid 0\}\}$	1	2
5	$1*$	$1*$	$\frac{5}{4} < V < 2$	1
6	$1*$	$1*$	$2*$	$1 < V < 2$
7	$1*$	$1*$	$\mid\mid\ 2$	
8	$1*$	$1*$	$\mid\mid\ 2$	

Figure 4.2: $\mathrm{T}^{a+1}\Box^b\mathrm{F}^a$, first part

$a\backslash b$	7	8	9	10	11	12	13	14	15	16
1	$6*$	7	$8*$	9	$10*$	11	$12*$	13	$14*$	15
2	$5*$	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{15}{2}$	$\frac{17}{2}$	$\frac{19}{2}$	$\frac{41}{4}$	$\frac{23}{2}$	$12*$	13
3	$\frac{15}{4}$	$\frac{9}{2}$	$< \frac{11}{2}$	< 6						
4	$\frac{5}{2} < V < 3$	< 4								
5	$\frac{5}{2} < V < 3$									
6	< 2 and $\mid\mid\ 3$									
7										

Figure 4.3: $\mathrm{T}^{a+1}\Box^b\mathrm{F}^a$, second part

$a\backslash b$	3	4	5	6	7	8	9	10	11	12
1	$4*$	6	$8*$	10	$12*$	14	$16*$	18	$20*$	22
2	$2*$	$4*$	$\frac{95}{16}$	$\frac{15}{2}$	$9*$	$\{11 \mid 11*\}$	12	$\frac{29}{2}$	$15*$	$17*$
3	$\frac{3}{2}$	$\{\{\frac{5}{2} \mid 2\} \mid\mid 2\}$	4	$\{\frac{11}{2} \mid \frac{41}{8}\}$	7					
4	$2*$	$\{\{4* \mid 2*\} \mid\mid \{\frac{3}{2} \mid 1\}\}$	3	4	$5*$					
5	$2*$	$2*$	$3 < V < 4$	3	5					
6	$2*$	$2*$	$\mid\mid\ 4$							
7	$2*$	$2*$	< 3							

Figure 4.4: $\mathrm{T}^{a+2}\Box^b\mathrm{F}^a$

$a\backslash b$	3	4	5	6	7	8
1	$6*$	9	$12*$	15	$18*$	21
2	$3*$	$\{6 \mid \frac{11}{2}\})$	$\{\frac{17}{2} \mid 8\})$	$11*$	13	$\frac{31}{2}$
3	$\frac{5}{2}$	L	$\frac{41}{8}$	$\|\|\ 8$		
4	$3*$	$\frac{5}{2}$	5	$\|\|\ 5$		
5	$3*$	$\|\|\ 3$	$5 < V < 6$			
6	$3*$	$3*$				

Figure 4.5: $\mathrm{T}^{a+3}\square^{b}\mathrm{F}^{a}$

Note 1) $||G$ means "can not be compared to G".
Note 2) We drop the values of the first two columns where $b = 1, 2$ since they will all be proved in the appendix.
Note 3) L means long. We are not writing it out here.

Erickson's conjecture 4 is false since $\mathrm{T}^{7}\square^{7}\mathrm{F}^{6} > 2$.

(**Erickson's conjecture 4**: $\mathrm{T}^{a}\square^{a}\mathrm{F}^{a-1} = 1$ or $\{1 \mid 1\}$ for all $a \geq 1$.)

We believe that there are no patterns in a for positions of the form $\mathrm{T}^{a+k}\square^{a+l}\mathrm{F}^{a}$; for any fixed $k \geq 1, l \geq 0$.

4.4 New Conjectures and Future Work

In [4], Jeff Erickson made 6 conjectures. Jesse Hull proved conjecture 6 (Toads and Frogs is NP-hard) in 2000. In this thesis I proved conjecture 1 (in collaboration with Zeilberger) (next chapter), 2 (previous chapter), 3 (next chapter) and disproved conjecture 4. Conjecture 5 is still open. We restate conjecture 5 here.

Erickson's conjecture 5:

$\mathrm{T}^{a}\square^{b}\mathrm{F}^{a}$ is an infinitesimal for all a,b except (a,b) =(3,2)

This conjecture seems very interesting and hard but ,I think, not impossible to prove. We split Erickson's fifth conjecture into 2 stronger conjectures which are conjectures 3 and 4 here.

We believe that there are still a lot of nice patterns and conjectures in this game that we overlooked. Once RAM gets cheaper and Maple gets faster, we will have more information.

Conjecture 1) Assume $b \geq 0$, $a \geq 1$, $L \geq 0$ and $R \geq 0$

1.1) $$\square^R \mathrm{T}^a \square^b \mathrm{F} \square^R = \begin{cases} \{\{a-2 \mid 1\} \mid 0\} & \text{if } R=0 \text{ and } b=1 \\ (a-1)(b-1+R) & \text{if } b \text{ is even} \\ (a-1)(b-1+R)* & \text{if } b \text{ is odd and} (R,b) \neq (0,1) \end{cases}$$

1.2) For $R \geq 1$, $$\square^{R-1} \mathrm{T}^a \square^b \mathrm{F} \square^R = \begin{cases} (a-1)(b-1+R) & \text{if } b \text{ is even} \\ 1/2 + (a-1)(b-1+R) & \text{if } b \text{ is odd} \end{cases}$$

1.3) For $R - L \geq 2, \square^L \mathrm{T}^a \square^b \mathrm{F} \square^R = (R-L-1) + (a-1)(b-1+R)$

Conjecture 2) For $a \geq 7$, $$\mathrm{TT}\square^a\mathrm{FF} = \begin{cases} * & \text{when } a = 7+6n, \quad n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture 3) $\mathrm{T}^a \square^b \mathrm{F}^a = *$ for any $a > b > 0$, except for $b = 2$.

Conjecture 4) $\mathrm{T}^a \square^b \mathrm{F}^a = 0$ or $*$ for any $b \geq a > 0$.

Conjecture 5) For a fixed integer $C \geq 3$, $\exists a_0$ such that $\mathrm{T}^C \square^a \mathrm{F}^C = 0$ for all $a \geq a_0$.

Future Work

1) Categorize all the positions that have exactly one Frog (general class B1) (conjecture 1 might be a good start).

Chapter 5

More Values of positions in "Toads and Frogs"

5.1 Introduction

In this chapter we prove the values of four infinite families of starting positions, three of which could not be solved by Symbolic Finite-state method. All four positions have beautiful values. This shows that the patterns of the values of the game "Toads and Frogs" are not only restricted to the classes Aij or Bij but also for the general class Ai.

The proofs in this chapter are tedious. But in the future, we hope to have a new method (hopefully along the same lines as the Symbolic Finite-state method) for making the proofs more automatic or at least shorten them.

5.2 Lemma and Convention

We will refer to the lemma below a lot. We state it here.

Lemma 5.2.1. ***One side Death Leap Principle*** *(One side DLP): if X is the position where the only possible move of Left is a jump and there is no two or more consecutive empty square in X then* $X \leq 0$.

Proof We have to show that when Left moves first and two players take turn playing, Left will lost(Left will run out of the legal move first). This is true since after Left jumps over one of the F, Right can response by moving to the empty square where the F was jumped over.

Example 1)TTF□TTF□F ≤ 0.

Example 2) TTTF□F□TF ≤ 0.

Convention:

In the following sections of the appendix, we will prove the positions using the "shorthand" notation. We explain by example.

Example: To show: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{FT}\square\mathrm{F}^b \leq \frac{1}{2}$; $k \geq 0, a \geq 0, b \geq 1$.

We will have to show $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{FT}\square\mathrm{F}^b - \frac{1}{2} \leq 0$.

That is to show $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{FT}\square\mathrm{F}^b - \{0 \mid 1\} \leq 0$.

To show $G \leq 0$ we need to show that when Left moves first and two players take turn playing, Right will win. (On the other hand, to show $G \geq 0$ we need to show that when Right moves first and two players take turn playing, Left will win.)

We will show that in these two sum games, for all the possible choices of Left moves, Right can find a response to the move so that he will win at the end.

We will do some case analysis here. In the above position Left has three choices. In the proof we will see

$$\overset{\overset{2}{\rightarrow}}{\mathrm{T}^a}\ \mathrm{F}\square\mathrm{T}^k\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^b \leq \overset{\overset{3}{\leftarrow}}{\tfrac{1}{2}}\ .$$

We will write the response of Right immediately.

Case 1: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{F}\square\mathrm{T}\mathrm{F}^b \leq 0$.

Explanation: Right responds by picking the left option of $\{0 \mid 1\}$ on the right hand side.

Case 2: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}^{k+1}\mathrm{F}\mathrm{T}\square\mathrm{F}^b \leq \frac{1}{2}$.

Explanation: Right responds by moving the left most F.

Case 3: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{F}\mathrm{T}\mathrm{F}\square\mathrm{F}^{b-1} \leq 1$.

Explanation: Left picks the right option of $\{0 \mid 1\}$ on the right hand side. Right responds by moving the rightmost F.

Note:

1) When the position simplifies to one of the known ones of chapter 3 (which are mostly positions in class A), we will claim such results without reproving them.

2) The positions we considered in Toads and Frogs are "hot", which means that the players select a good move and fight for an advantage. We will not consider the possible moves in a cold game which is a whole integer.

5.3 $\mathrm{T}^a\square\square\mathrm{F}^a$, $a \geq 4$

We show $\mathrm{T}^a\square\square\mathrm{F}^a$ is an infinitesimal , $a \geq 4$. The observation comes from table 2 of chapter 4 when $b = 2$.

Lemma 5.3.1. *For any fixed integer* $n \geq 3$, $\widetilde{\mathrm{L}}\ \overset{\overset{2}{\rightarrow}}{\mathrm{T}^a}\ \mathrm{F}\square\mathrm{T}^k\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^b \ \overset{\overset{3}{\leftarrow}}{\leq} \frac{1}{2^n}$, $k \geq 0, a \geq 0, b \geq 1$.

Prove: By induction on a.

Base Case: $a = 0$, $\square T^k F \overset{\xrightarrow{1}}{T} \square F^b \overset{\xleftarrow{2}}{\leq} \frac{1}{2^n}$.

Case 1: $\square T^k F \square T F^b \leq 0$, true by one side DLP.

Case 2: $\square T^k F T F \square F^{b-1} \leq \frac{2}{2^n}$. The left hand side is ≤ 0 by one side DLP.

Induction Step: $\widetilde{L} \overset{\xrightarrow{2}}{T^a} F \square T^k F \overset{\xrightarrow{1}}{T} \square F^b \overset{\xleftarrow{3}}{\leq} \frac{1}{2^n}$.

Case 1: $\widetilde{L} T^a F \square T^k F \square T F^b \leq 0$, true by one side DLP.

Case 2: $\widetilde{L} T^{a-1} F \square T^{k+1} F T \square F^b \leq \frac{1}{2^n}$, true by induction.

Case 3: $\widetilde{L} T^a F \square T^k F T F \square F^{b-1} \leq \frac{2}{2^n}$. The left hand side is ≤ 0 by one side DLP.

Theorem 5.3.1. $T^a \square\square F^a$ *is an infinitesimal* , $a \geq 4$.

By symmetry we only need to show:

For any fixed integer $n \geq 3$, $T^a \square\square F^a \leq \frac{1}{2^n}$, $a \geq 4$.

$\overset{\xrightarrow{I}}{T^a} \square\square F^a \overset{\xleftarrow{II}}{\leq} \frac{1}{2^n}$.

I) $\overset{\xrightarrow{2}}{T^{a-1}} \square \overset{\xrightarrow{1}}{T} F \square F^{a-1} \overset{\xleftarrow{3}}{\leq} \frac{1}{2^n}$

II) $\overset{\xrightarrow{1}}{T^a} \square F \square F^{a-1} \overset{\xleftarrow{2}}{\leq} \frac{2}{2^n}$

I) Case 1: $\overset{\xrightarrow{1}}{T^{a-1}} \square F \square T F^{a-1} \overset{\xleftarrow{2}}{\leq} \frac{1}{2^n}$

Case 1.1: $T^{a-2} F \overset{\xrightarrow{1}}{T} \square\square T F^{a-1} \overset{\xleftarrow{2}}{\leq} \frac{1}{2^n}$

Case 1.1.1: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TFT}\square\mathrm{F}^{a-2} \leq \frac{1}{2^n}$, true by lemma C.1.

Case 1.1.2: $\mathrm{T}^{a-2}\mathrm{F}\ \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^{a-2} \leq \overset{\overset{3}{\leftarrow}}{\frac{2}{2^n}}$

Case 1.1.2.1: $\mathrm{T}^{a-2}\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\square\mathrm{TF}^{a-2} \leq \overset{\overset{2}{\leftarrow}}{\frac{2}{2^n}}$

Case 1.1.2.1.1: $\square\mathrm{T}\square\mathrm{TF}^{a-2} \leq \frac{2}{2^n}$.

The left hand side is $\{0 \,||\, \{0 \mid \{-1 \mid 5-a\}\}\}$.

Case 1.1.2.1.2: $\mathrm{T}^{a-2}\mathrm{FTF}\square\mathrm{FT}\square\mathrm{F}^{a-3} \leq \frac{4}{2^n}$, true by lemma C.1.

Case 1.1.2.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TFTF}\square\mathrm{F}^{a-3} \leq \frac{2}{2^n}$.

The left hand side is ≤ 0 by one side DLP.

Case 1.1.2.3: $\mathrm{T}^{a-2}\mathrm{F}\ \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{F}^{a-3} \leq \overset{\overset{3}{\leftarrow}}{\frac{4}{2^n}}$

Case 1.1.2.3.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\mathrm{F}\square\mathrm{FTF}^{a-3} \leq 0$

$$\Rightarrow \mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TFF}\square\mathrm{TF}^{a-3} \leq 0$$

The statement is true by one side DLP.

Case 1.1.2.3.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TFTFF}\square\mathrm{F}^{a-4} \leq \frac{4}{2^n}$.

The left hand side is ≤ 0 by one side DLP.

Case 1.1.2.3.3: $\mathrm{T}^{a-2}\mathrm{FTF}\square\mathrm{TF}\square\mathrm{F}^{a-3} \leq \frac{8}{2^n}$.

The left hand side is ≤ 0 by one side DLP.

Case 1.2: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\mathrm{FF}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^{a-2} \leq \overset{\overset{3}{\leftarrow}}{\frac{2}{2^n}}$

Case 1.2.1: $\mathrm{T}^{a-1}\mathrm{F}\Box\mathrm{F}\Box\mathrm{TF}^{a-2} \leq \frac{2}{2^n}$.

The left hand side is ≤ 0 by one side DLP.

Case 1.2.2: $\mathrm{T}^{a-2}\mathrm{FT}\Box\mathrm{FT}\Box\mathrm{F}^{a-2} \leq \frac{2}{2^n}$. This is the case 1.1.2

Case 1.2.3: $\mathrm{T}^{a-1}\mathrm{F}\Box\mathrm{FT}\Box\mathrm{F}^{a-2} \leq \frac{4}{2^n}$, true by lemma C.1.

Case 2: $\mathrm{T}^{a-2}\Box\mathrm{TTFF}\Box\mathrm{F}^{a-2} \leq \frac{1}{2^n}$. The left hand side is 0.

Case 3: $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a-1}}\Box\mathrm{TFF}\Box\mathrm{F}^{a-2} \leq \overset{\overset{2}{\leftarrow}}{\frac{2}{2^n}}$

Case 3.1: $\mathrm{T}^{a-2}\Box\mathrm{TTFF}\Box\mathrm{F}^{a-2} \leq 0$. This is clearly true.

Case 3.2: $\mathrm{T}^{a-1}\mathrm{FT}\Box\mathrm{F}\Box\mathrm{F}^{a-2} \leq \frac{4}{2^n}$. The left hand side is ≤ 0.

II) Case 1: $\mathrm{T}^{a-1}\Box\mathrm{TFF}\Box\mathrm{F}^{a-2} \leq \frac{2}{2^n}$. This is I) case 3.

Case2: $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a}}\Box\mathrm{FF}\Box\mathrm{F}^{a-2} \leq \overset{\overset{2}{\leftarrow}}{\frac{4}{2^n}}$

Case 2.1: $\mathrm{T}^{a-1}\mathrm{FT}\Box\mathrm{F}\Box\mathrm{F}^{a-2} \leq \frac{4}{2^n}$. This is I) case 3.2.

Case 2.2: $\mathrm{T}^{a}\mathrm{F}\Box\mathrm{F}\Box\mathrm{F}^{a-2} \leq \frac{8}{2^n}$. The left hand side is ≤ 0 by one side DLP.

The theorem is proved. $\Box$

5.4 $\mathrm{T}^a\square\square\square\mathrm{FFF}$, $a \geq 5$

We show $\mathrm{T}^a\square\square\square\mathrm{FFF} = a - \frac{7}{2}$, $a \geq 5$.

We are supposed to be able to prove the values of the class $\square\square\square\mathrm{FFF}$ by a Symbolic Finite-State method. Then the results above would follow as special cases of many positions in the class. But as we mentioned in chapter 4, we could not get the computer to handle classes with three blanks and three frogs yet. It would takes days for a human to do make such conjectures by hand. For now we prove the value of $\mathrm{T}^a\square\square\square\mathrm{FFF}$, $a \geq 5$, which is Jeff Erickson's conjecture 3, by hand. The proof is however assisted by the Maple program written by the author.

Below is the outline.

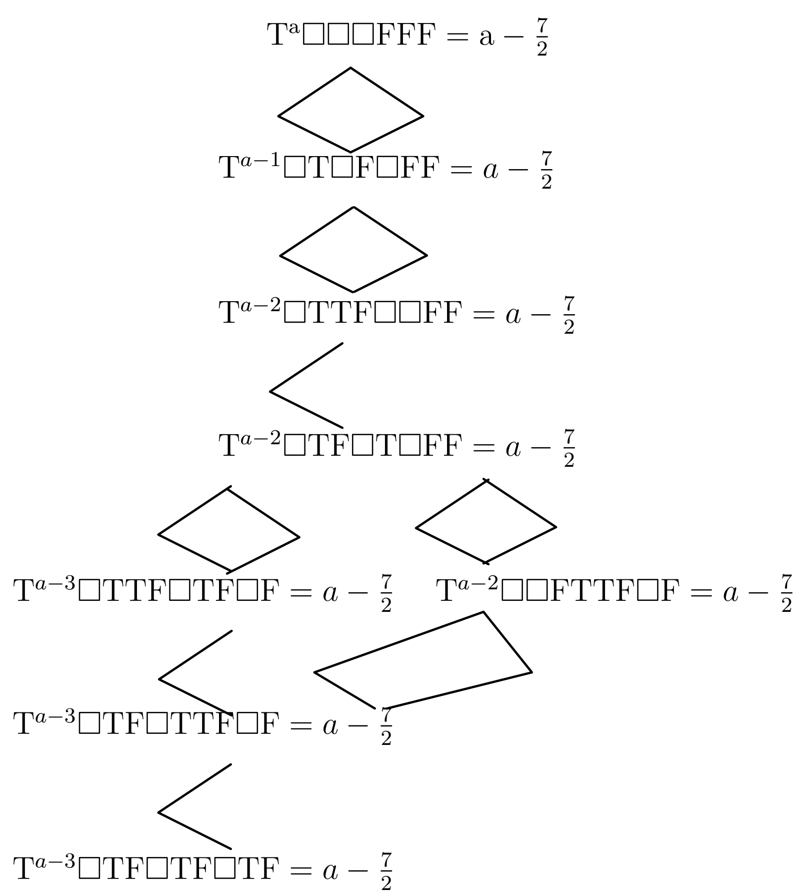


Figure 5.1: Main tree

We will work from the bottom of the main tree (figure 1) and work our way up.

Lemma 5.4.1. $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} = a - \frac{7}{2}$, $a \geq 5$.

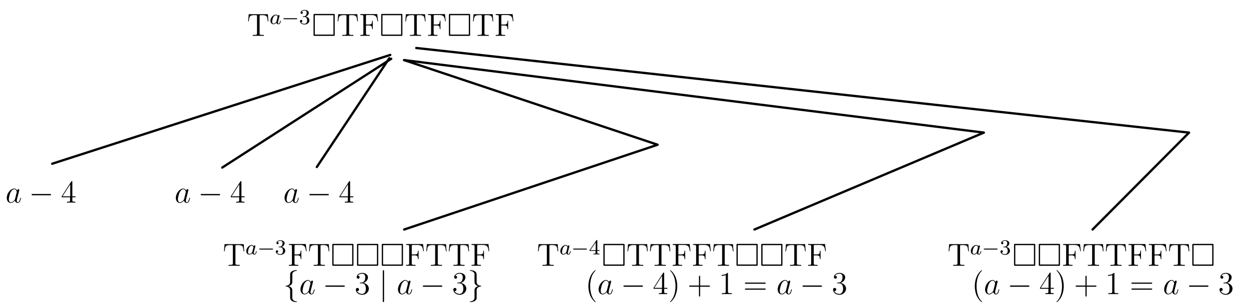


Figure 5.2: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} = a - \frac{7}{2}$

Left options.

On the left hand side, $\mathrm{T}^{a-3}\square\mathrm{TF}\square\square\mathrm{FTTF} = \mathrm{T}^{a-3}\square\mathrm{TF}\square\square\mathrm{F} = a-4$, $a \geq 4$. We know this values, chapter 3, from Symbolic Finite-state method we developed in chapter 2.

I need to show that the other options on the left hand side got dominated by this option.

L1: To Show $\mathrm{T}^{a-4}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{TF} \leq a - 4$.

L2: To Show $\mathrm{T}^{a-3}\square\square\mathrm{FTTF}\square\mathrm{TF} \leq a - 4$.

Right options.

R1: To Show $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} \leq a - 3$.

R2: To Show $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} \leq \{a - 3 \mid a - 3\}$.

We see, from the picture, the three positions of the right side are $\{a-3 \mid a-3\}, a-3$ and $a-3$. We will show the statements R1 and R2 are true. Then by applying bypass reversible move rule, we will have the value of the right options is $a-3$.

Then we can conclude that $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} = \{a-4 \mid a-3\} = a-\frac{7}{2}$.

Below are the proofs of statements **L1, L2, R1, R2**.

L1: To Show $\overset{\xrightarrow{3}}{\mathrm{T}^{a-4}}\ \square\mathrm{T}\ \overset{\xrightarrow{2}}{\mathrm{T}}\ \mathrm{F}\square\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{TF} \leq a-4$.

Case1: $\mathrm{T}^{a-4}\square\mathrm{TTF}\square\mathrm{F}\square \leq a-4$. The left hand side is $\{\{a-4 \mid a-4\} \mid a-4\}$.

Case 2: $\overset{\xrightarrow{3}}{\mathrm{T}^{a-4}}\ \square\ \overset{\xrightarrow{2}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{T}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{TF} \leq a-4$.

Case 2.1: $\mathrm{T}^{a-4}\square\mathrm{TF}\square\mathrm{TF}\square \leq a-4$. The left hand side is $a-4$.

Case 2.2: $\mathrm{T}^{a-4}\square\square\mathrm{FTTTFFT}\square \leq a-4$. The left hand side is $(a-5)+1$.

Case 2.3: $\mathrm{T}^{a-5}\square\mathrm{TTF}\square\mathrm{TTFFT}\square \leq a-4$. The left hand side is $(a-5)+1$.

Case 3: $\mathrm{T}^{a-5}\square\mathrm{TTTFFT}\square\square\mathrm{TF} \leq a-4$. The left hand side is $(a-5)+1$.

L2: To Show $\overset{\xrightarrow{2}}{\mathrm{T}^{a-3}}\ \square\square\mathrm{FT}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{TF} \leq a-4$.

Case1: $\mathrm{T}^{a-3}\square\square\mathrm{FTF}\square \leq a-4$. The left hand side already is $a-4$.

Case 2: $\mathrm{T}^{a-4}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{TF} \leq a-4$, this is the case 2 of **L1**.

R1: To Show $\overset{\overset{3}{\rightarrow}}{T^{a-3}}\,\square\,\overset{\overset{2}{\rightarrow}}{T}\,F\square\,\overset{\overset{1}{\rightarrow}}{T}\,F\square TF \le a-3$.

Case1: $T^{a-3}\square TF\square\square F \le a-4$. The left hand side is $a-4$.

Case 2: $T^{a-3}\square\square FTTFFT\square \le a-3$. The left hand side is $(a-4)+1$.

Case 3: $T^{a-4}\square TTFFT\square\square TF \le a-3$. The left hand side is $(a-4)+1$.

R2: To Show $\overset{\overset{3}{\rightarrow}}{T^{a-3}}\,\square\,\overset{\overset{2}{\rightarrow}}{T}\,F\square\,\overset{\overset{1}{\rightarrow}}{T}\,F\square TF \le \{a-3\ |\overset{\overset{4}{\leftarrow}}{a-3}\}$.

Case 1: $T^{a-3}\square TF\square\square F \le a-3$. The left hand side is $a-4$.

Case 2: $T^{a-3}\square\square FTTF\square TF \le a-3$. The left hand side is $\le a-4$ by **L2**.

Case 3: $T^{a-4}\square TTF\square TF\square TF \le a-3$. The left hand side is $\le a-4$ by **L1**.

Case4: $T^{a-3}F\,\overset{\overset{2}{\rightarrow}}{T}\,\square\square\,\overset{\overset{1}{\rightarrow}}{T}\,F\square TF \le a-3$.

Case 4.1: $T^{a-3}FT\square\square F\square \le a-3$. The left hand side is $\{\{a-3 \mid a-3\} \mid a-3\}$.

Case 4.2: $\overset{\overset{2}{\rightarrow}}{T^{a-3}}\,F\square TF\,\overset{\overset{1}{\rightarrow}}{T}\,\square\square TF \le a-3$.

Case 4.2.1: $T^{a-3}FFT\square\square T\square TF \le a-3$. The left hand side is $\{3 \mid 3\}$ which assure the statement for $a \ge 7$.

Case 4.2.2: $T^{a-4}F\square TTFT\square\square TF \le a-3$.

$$\Rightarrow \mathrm{T}^{a-2}\mathrm{FT}\square\square\mathrm{TF} \leq a-3 \text{ , The left hand side is 1.}$$

We now have lemma 2.1. We are now one step closer to the main theorem. We now move up the picture to prove another statement.

Lemma 5.4.2. $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} = a - \frac{7}{2}$, $a \geq 5$.

Need to show:

I) $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} \leq a - \frac{7}{2}$.

II) $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{TF} \geq a - \frac{7}{2}$.

I) To Show $\overset{\xrightarrow{3}}{\mathrm{T}^{a-3}} \square \overset{\xrightarrow{2}}{\mathrm{T}} \mathrm{F}\square\mathrm{T} \overset{\xrightarrow{1}}{\mathrm{T}} \mathrm{F}\square\mathrm{F} \leq \overset{\xleftarrow{4}}{a} - \frac{7}{2}$.

Case1: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} \leq a - \frac{7}{2}$. This is true by lemma 2.1.

Case 2: $\mathrm{T}^{a-3}\square\square\mathrm{FTTTFF}\square \leq a - \frac{7}{2}$. The left hand side is $a-4$.

Case 3: $\mathrm{T}^{a-4}\square\mathrm{TTF}\square \leq a - \frac{7}{2}$. The left hand side is $a-4$.

Case 4: $\mathrm{T}^{a-3}\square\mathrm{TF}\square \leq a-3$. The left hand side is $a-3$.

II) To Show $\mathrm{T}^{a-3}\square\mathrm{T} \overset{\xleftarrow{1}}{\mathrm{F}} \square\mathrm{TTF}\square \overset{\xleftarrow{2}}{\mathrm{F}} \geq \overset{\xrightarrow{3}}{a} - \frac{7}{2}$.

Case1: $\mathrm{T}^{a-3}\mathrm{F}\square\mathrm{T}\square\mathrm{TTF}\square\mathrm{F} \geq a - \frac{7}{2}$. The left hand side is $\geq 1 + (a-3) - 1 = a-3$.

Case 2: $\mathrm{T}^{a-3}\square\mathrm{TF}\square \geq a-3$. The left hand side is $a-3$.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{T}\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\square\mathrm{T}\square\overset{\overset{2}{\leftarrow}}{\mathrm{F}}\mathrm{TF} \geq \quad a-4$.

Case 3.1: $\mathrm{T}^{a-3}\mathrm{F}\square\mathrm{T}\square\mathrm{T}\square\mathrm{FTF} \geq a-4$.

The left hand side is $\geq 1+(a-3)-1 = a-3$.

Case 3.2: $a-4 \geq a-4$. by lemma 2.1.

We prove lemma 2.2 here.

Lemma 5.4.3. $\mathrm{T}^{a-2}\square\square\mathrm{FTTF}\square\mathrm{F} = a-\frac{7}{2}$, $a \geq 5$.

Need to show :

I) $\mathrm{T}^{a-2}\square\square\mathrm{FTTF}\square\mathrm{F} \leq a-\frac{7}{2}$.

II) $\mathrm{T}^{a-2}\square\square\mathrm{FTTF}\square\mathrm{F} \geq a-\frac{7}{2}$.

I) To Show $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\square\square\mathrm{FT}\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\mathrm{F}\square\mathrm{F} \leq \overset{\overset{3}{\leftarrow}}{a}-\frac{7}{2}$.

Case1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\square\square\mathrm{F}\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\mathrm{F}\square\mathrm{TF} \leq \overset{\overset{3}{\leftarrow}}{a}-\frac{7}{2}$.

Case 1.1: $\mathrm{T}^{a-2}\square\square\mathrm{F}\square\mathrm{F} \leq a-4$. The left hand side is $a-4$.

Case 1.2: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} \leq a-\frac{7}{2}$. This is true by lemma 2.1.

Case 1.3: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\square\mathrm{F}\square\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\mathrm{F}\square\mathrm{TF} \leq a-3$.

Case 1.3.1: $\mathrm{T}^{a-2}\mathrm{F}\square\square\square\mathrm{F} \leq a-3$. The left hand side is $a-3$.

Case 1.3.2: $a-3 \leq a-3$, by lemma 2.1.

Case 2: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} \leq a-\frac{7}{2}$. The left hand side is $a-\frac{7}{2}$ by lemma 2.2.

Case 3: $\mathrm{T}^{a-2}\square\square\mathrm{FTTFF}\square \leq a-3$. The left hand side is $a-3$.

II) To Show $\mathrm{T}^{a-2}\square\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TTF}\square \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq \overset{\overset{3}{\rightarrow}}{a-\frac{7}{2}}$.

Case1: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} \geq a-\frac{7}{2}$. The left hand side is $a-\frac{7}{2}$ by lemma 2.2.

Case 2: $\mathrm{T}^{a-2}\square\square\mathrm{F} \geq a-3$. The left hand side is $a-3$.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{T}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TTF}\square \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq a-4$.

Case 3.1: $a-4 \geq a-4$, by lemma 2.2.

Case 3.2: $\mathrm{T}^{a-3}\square\square\mathrm{TF} \geq a-4$. The left hand side is $a-3$.

We have lemma 2.3 here. Lemma 2.4 is similar to lemma 2.3. They are also at the same level in the picture.

Lemma 5.4.4. $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} = a-\frac{7}{2}$, $a \geq 5$.

Need to show :

I) $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} \leq a-\frac{7}{2}$.

II) $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} \geq a - \frac{7}{2}$.

I) To Show $\overset{\overset{3}{\rightarrow}}{\mathrm{T}^{a-3}}\ \square\mathrm{T}\ \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{F} \leq\ \overset{\overset{4}{\leftarrow}}{a} - \frac{7}{2}$.

Case1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-3}}\ \square\mathrm{T}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{F}\square\mathrm{TF} \leq\ \overset{\overset{3}{\leftarrow}}{a} - \frac{7}{2}$.

Case1.1: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} \leq a - \frac{7}{2}$. This is true by lemma 2.1.

Case1.2: $\mathrm{T}^{a-4}\square\mathrm{TTTFF}\square\square\mathrm{TF} \leq a - \frac{7}{2}$. The left hand side is $(a-4)-1$.

Case1.3: $\mathrm{T}^{a-3}\square\mathrm{TTFF}\square\square\mathrm{TF} \leq a - 3$. The left hand side is $(a-3)-1$.

Case 2: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} \leq a - \frac{7}{2}$. The left hand side is $a - \frac{7}{2}$ by lemma 2.2.

Case 3: $\mathrm{T}^{a-4}\square\mathrm{TTTFFT}\square\square\mathrm{F} \leq a - \frac{7}{2}$. The left hand side is $a-4$.

Case 4: $\mathrm{T}^{a-3}\square\mathrm{TTFFT}\square\square\mathrm{F} \leq a - 3$. The left hand side is $a-3$.

II) To Show $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{T}\ \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \square\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq\ \overset{\overset{3}{\rightarrow}}{a} - \frac{7}{2}$.

Case1: $\mathrm{T}^{a-3}\square\mathrm{TTFFT}\square\square\mathrm{F} \geq a - 3$. The left hand side is $a-3$.

Case 2: $\mathrm{T}^{a-3}\square\mathrm{T}\square\mathrm{FTTFF}\square \geq a - \frac{7}{2}$. The left hand side is $\{a-3 \mid a-3\}$.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{T}\square\ \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TTF}\square\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq a - 4$.

Case 3.1: $a - 4 \geq a - 4$, by lemma 2.2.

Case 3.2: $\mathrm{T}^{a-3}\square\square\mathrm{TF} \geq a - 4$. The left hand side is $a-3$.

Lemma 5.4.5. $\mathrm{T}^{a-2}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{FF} = a - \frac{7}{2}$, $a \geq 5$.

Need to show :

I) $\mathrm{T}^{a-2}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{FF} \leq a - \frac{7}{2}$.

II)) $\mathrm{T}^{a-2}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{FF} \geq a - \frac{7}{2}$.

I) To Show $\overset{\overset{3}{\rightarrow}}{\mathrm{T}^{a-2}} \square \overset{\overset{2}{\rightarrow}}{\mathrm{T}} \mathrm{F}\square \overset{\overset{1}{\rightarrow}}{\mathrm{T}} \square\mathrm{FF} \leq \overset{\overset{4}{\leftarrow}}{a} - \frac{7}{2}$.

Case1: $\mathrm{T}^{a-2}\mathrm{F} \overset{\overset{1}{\rightarrow}}{\mathrm{T}} \square\square\square\mathrm{TFF} \leq a \overset{\overset{2}{\leftarrow}}{-} \frac{7}{2}$.

Case 1.1: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{T}\square\mathrm{FT}\square\mathrm{F} \leq a - \frac{7}{2}$.

The left hand side goes to $\Rightarrow \mathrm{T}^{a-1}\square\mathrm{FT}\square\mathrm{F} = \{1 \mid 1\}$.

Case 1.2: $\mathrm{T}^{a-2}\mathrm{F} \overset{\overset{2}{\rightarrow}}{\mathrm{T}} \square\square\mathrm{F} \overset{\overset{1}{\rightarrow}}{\mathrm{T}} \square\mathrm{F} \leq a - 3$.

Case 1.2.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\mathrm{F}\square\square\mathrm{TF} \leq a - 3$.

The left most position will get block eventually.

Case 1.2.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{F} \leq a - 3$.

The left hand side goes to $\Rightarrow \mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}\square\mathrm{F} \leq a - 3$.

The left hand side is $\{\{\{a - 3 \mid 2\} \mid 1\} \mid\mid 0\}$.

Case 2: $\mathrm{T}^{a-2}\square\square\mathrm{FTTF}\square\mathrm{F} \leq a - \frac{7}{2}$. The left hand side is $a - \frac{7}{2}$ by lemma 2.3.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} \leq a - \frac{7}{2}$. The left hand side is $a - \frac{7}{2}$ by lemma 2.4.

Case 4: $\overset{\overset{3}{\rightarrow}}{\mathrm{T}^{a-2}} \,\square\, \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\,\mathrm{F}\square\, \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{F} \leq a-3.$

Case 4.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\square\square\mathrm{FTF} \leq a-3.$

$\Rightarrow \mathrm{T}^{a-2}\mathrm{F}\square\mathrm{T}\square\mathrm{F}\square\mathrm{TF} \leq a-3.$

$\Rightarrow \mathrm{T}^{a-1}\square\mathrm{F}\square\mathrm{TF} \leq a-3.$, the left hand side is 1.

Case 4.2: $\mathrm{T}^{a-2}\square\square\mathrm{F} \leq a-3.$, the left hand side is $a-3$.

Case 4.3: $a-3 \leq a-3$. This is true by lemma 2.4.

II) To Show $\mathrm{T}^{a-2}\square\mathrm{T}\,\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\,\square\mathrm{T}\square\,\overset{\overset{2}{\leftarrow}}{\mathrm{F}}\,\mathrm{F} \geq \overset{\overset{3}{\rightarrow}}{a-}\frac{7}{2}$.

Case1: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{T}\square\mathrm{T}\square\,\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\,\mathrm{F} \geq \overset{\overset{2}{\rightarrow}}{a-}\frac{7}{2}$.

Case1.1: $\mathrm{T}^{a-2}\mathrm{F}\square\square\mathrm{TTF}\square\mathrm{F} \geq a-\frac{7}{2}$. The left hand side is $\geq (a-2)-1$.

Case1.2: $\mathrm{T}^{a-2}\square\square\mathrm{TT}\square\mathrm{FF} \geq a-4$. The left hand side is $\geq (a-2)-2$.

Case2: $\mathrm{T}^{a-2}\mathrm{F}\square\square\mathrm{FTTF}\square\mathrm{F} \geq a-\frac{7}{2}$. This is true by lemma 2.3.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{T}\square\,\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\,\mathrm{F} \geq \ a-4.$

$\Rightarrow a-4 \geq a-4$, by lemma 2.4.

Lemma 5.4.6. $\mathrm{T}^{a-2}\square\mathrm{TTF}\square\square\mathrm{FF} = a-\frac{7}{2}$, $a \geq 5$.

Need to show :

I) $T^{a-2}\square TTF\square\square FF \le a - \frac{7}{2}$.

II) $T^{a-2}\square TTF\square\square FF \ge a - \frac{7}{2}$.

I) To Show $\overset{\xrightarrow{2}}{T^{a-2}}\,\square T\overset{\xrightarrow{1}}{T}F\square\square FF \le \overset{\xleftarrow{3}}{a - \frac{7}{2}}$.

Case 1: $T^{a-2}\square TF\square T\square FF \le a - \frac{7}{2}$. This is true by lemma 2.5.

Case 2: $\overset{\xrightarrow{2}}{T^{a-3}}\,\square TT\overset{\xrightarrow{1}}{T}F\square F\square F \le \overset{\xleftarrow{3}}{a - \frac{7}{2}}$.

Case 2.1: $T^{a-3}\square TTF\square TF\square F \le a - \frac{7}{2}$. This is true by lemma 2.4.

Case 2.2: $T^{a-4}\square TTTTFF\square\square F \le a - \frac{7}{2}$. The left hand side is $(a-4)-2$.

Case 2.3: $T^{a-3}\square TTTFF\square\square F \le a - 3$. The left hand side is $(a-3)-2$.

Case 3: $\overset{\xrightarrow{2}}{T^{a-2}}\,\square T\overset{\xrightarrow{1}}{T}F\square F\square F \le a - 3$.

Case 3.1: $T^{a-2}\square TF\square TF\square F \le a - 3$. This is case I)4 of lemma 2.5.

Case 3.2: $T^{a-3}\square TTTFF\square\square F \le a - 3$. The left hand side is $(a-3)-2$.

II) To Show $T^{a-2}\square TTF\square\square\overset{\xleftarrow{1}}{F}F \ge \overset{\xrightarrow{2}}{a - \frac{7}{2}}$.

Case 1: $T^{a-2}\square T\square\overset{\xleftarrow{1}}{F}TF\square\overset{\xleftarrow{2}}{F} \ge \overset{\xrightarrow{3}}{a - \frac{7}{2}}$.

Case 1.1: $T^{a-3}\square TTF\square TF\square F \ge a - \frac{7}{2}$. This is true by lemma 2.4.

Case 1.2: $\mathrm{T}^{a-2}\square\square\mathrm{TFTFF}\square \geq a - \frac{7}{2}$.

The left hand side is $\geq \mathrm{T}^{a-2}\square\mathrm{F}\square = a - \frac{5}{2}$.

Case 1.3: $\mathrm{T}^{a-2}\square\square\mathrm{TFTF}\square\mathrm{F} \geq a - 4$.

The left hand side is $\geq \mathrm{T}^{a-2}\square\mathrm{F}\square - 1 = (a - \frac{5}{2}) - 1$.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{T}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{T}\square \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \mathrm{F} \geq \ \ a - 4$.

Case 2.1: $a - 4 \geq a - 4$, by lemma 2.5.

Case 2.2: $\mathrm{T}^{a-2}\square\square\mathrm{TFTF}\square\mathrm{F} \geq a - 4$. This is the same as case 1.3 above.

Lemma 5.4.7. $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{FF} = a - \frac{7}{2}$, $a \geq 5$.

Need to show :

I) $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{FF} \leq a - \frac{7}{2}$.

II) $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{FF} \geq a - \frac{7}{2}$.

I) To Show $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}\square\mathrm{FF} \leq\ \ a \overset{\overset{3}{\leftarrow}}{-} \frac{7}{2}$.

Case 1: $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\square\mathrm{TFF}\square\mathrm{F} \leq\ \ a \overset{\overset{2}{\leftarrow}}{-} \frac{7}{2}$.

Case 1.1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\ \square\mathrm{TF} \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}\square\mathrm{F} \leq\ \ a \overset{\overset{3}{\leftarrow}}{-} \frac{7}{2}$.

Case 1.1.1: $\mathrm{T}^{a-2}\mathrm{F} \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\ \square\square \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{F} \leq\ \ a \overset{\overset{3}{\leftarrow}}{-} \frac{7}{2}$. (The left hand side is $\{1 \mid \frac{1}{2}\}$)

Case 1.1.1.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\square\square\mathrm{FTF} \leq a - 4$.

$\Rightarrow \text{T}^{a-2}\text{F}\square\text{T}\square\text{F}\square\text{TF} \leq a-4.$

$\Rightarrow \text{T}^{a-1}\square\text{F}\square\text{TF} \leq a-4$. The left hand side is 1.

Case 1.1.1.2: $\text{T}^{a-2}\text{F}\square\text{TFT}\square\square\text{F} \leq a-\frac{7}{2}$.

$\Rightarrow \text{T}^{a-1}\text{FT}\square\square\text{F} \leq a-\frac{7}{2}$. The left hand side is 0.

Case 1.1.1.3: $\text{T}^{a-2}\text{FT}\square\text{FT}\square\square\text{F} \leq a-3$. The left hand side is ≤ 2.

Case 1.1.2: $\text{T}^{a-3}\square\text{TTFTF}\square\square\text{F} \leq a-\frac{7}{2}$.

The left hand side is $\leq (a-3)+\square\text{T}\square\text{F} = (a-3)-\frac{1}{2}$

Case 1.1.3: $\text{T}^{a-2}\text{F}\ \overset{\overset{2}{\rightarrow}}{\text{T}}\ \square\ \overset{\overset{1}{\rightarrow}}{\text{T}}\ \square\text{F}\square\text{F} \leq a-3.$

Case 1.1.3.1: $\text{T}^{a-2}\text{FT}\square\text{FT}\square\square\text{F} \leq a-3$. This is the same as case 1.1.1.3 above.

Case 1.1.3.2: $\text{T}^{a-2}\text{F}\square\text{TTF}\square\square\text{F} \leq a-3$.

$\Rightarrow$ The left hand side is $\leq \text{T}^{a}\text{F}\square\square\text{F} = \{\{1 \mid 1\} \mid\mid 0\}$.

Case 1.2: $\overset{\overset{2}{\rightarrow}}{\text{T}^{a-1}}\ \square\text{F}\ \overset{\overset{1}{\rightarrow}}{\text{T}}\ \square\text{F}\square\text{F} \leq a-3.$

Case 1.2.1: $\overset{\overset{2}{\rightarrow}}{\text{T}^{a-1}}\ \square\text{FF}\ \overset{\overset{1}{\rightarrow}}{\text{T}}\ \square\square\text{F} \leq a-3.$

Case 1.2.1.1: $\overset{\overset{2}{\rightarrow}}{\text{T}^{a-1}}\ \text{F}\square\text{F}\square\ \overset{\overset{1}{\rightarrow}}{\text{T}}\ \square\text{F} \leq a-3.$

Case 1.2.1.1.1: $\text{T}^{a-1}\text{FF}\square\square\square\text{TF} \leq a-3$. The left hand side is -2.

Case 1.2.1.1.2: The left hand side goes to $\Rightarrow \mathrm{T}^{a-1}\mathrm{F}\Box\mathrm{T}\Box\mathrm{F} \leq a-3$.

The left hand side is $\{\{\{a-3 \mid 2\} \mid 1\} \mid\mid 0\}$.

Case 1.2.1.2: $\mathrm{T}^{a-2}\mathrm{F} \overset{\overset{2}{\rightarrow}}{\mathrm{T}} \Box\mathrm{F} \overset{\overset{1}{\rightarrow}}{\mathrm{T}} \Box\Box\mathrm{F} \leq a-3$.

Case 1.2.1.2.1: $\mathrm{T}^{a-2}\mathrm{FTF}\Box\Box\mathrm{T}\Box\mathrm{F} \leq a-3$.

$\Rightarrow \Box\mathrm{T}\Box\mathrm{T}\Box\mathrm{F} = \{1 \mid 1\}$.

Case 1.2.1.2.2: $\mathrm{T}\Box\mathrm{T}\Box\Box\mathrm{F} \leq a-3$. The left hand side is 2.

Case 1.2.2: $\mathrm{T}^{a-2}\mathrm{FT}\Box\mathrm{T}\Box\mathrm{F}\Box\mathrm{F} \leq a-3$. This is case 1.1.3 above.

Case 2: $\mathrm{T}^{a-2}\Box\mathrm{TTF}\Box\Box\mathrm{FF} \leq a-\frac{7}{2}$. This is true by lemma 2.6.

Case 3: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}} \Box \overset{\overset{1}{\rightarrow}}{\mathrm{T}} \mathrm{F}\Box\Box\mathrm{FF} \leq a-3$.

Case 3.1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}} \Box\mathrm{F}\Box \overset{\overset{1}{\rightarrow}}{\mathrm{T}} \Box\mathrm{FF} \leq a-3$.

Case 3.1.1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}} \Box\mathrm{F}\Box\mathrm{F} \overset{\overset{1}{\rightarrow}}{\mathrm{T}} \Box\mathrm{F} \leq a-3$.

Case 3.1.1.1: $\mathrm{T}^{a-1}\mathrm{F}\Box\Box\mathrm{F}\Box\mathrm{TF} \leq a-3$.

$\Rightarrow \mathrm{T}^{a-1}\Box\mathrm{F}\Box\mathrm{TF} = 1$.

Case 3.1.1.2: $\mathrm{T}^{a-2}\mathrm{F} \overset{\overset{2}{\rightarrow}}{\mathrm{T}} \Box\Box\mathrm{F} \overset{\overset{1}{\rightarrow}}{\mathrm{T}} \Box\mathrm{F} \leq a-3$.

Case 3.1.1.2.1: $\mathrm{T}^{a-2}\mathrm{FT}\Box\mathrm{F}\Box\Box\mathrm{TF} \leq a-3$.

The statement above is true since the left most part of the position gets block eventually.

Case 3.1.1.2.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{F} \leq a-3$.

The left hand side goes to $\Rightarrow \mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}\square\mathrm{F} = \{\{a-3 \mid 2\} \mid 1\} \mid\mid 0\}$

Case 3.1.2: $a-3 \leq a-3$, by lemma 2.5.

Case 3.2: $a-3 \leq a-3$, by lemma 2.6.

II) To Show $\mathrm{T}^{a-1}\square\mathrm{T}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}} \square \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \mathrm{F} \geq a \overset{\overset{3}{\rightarrow}}{-} \frac{7}{2}$.

Case 1: $\mathrm{T}^{a-2}\square\mathrm{TTF}\square\square\mathrm{FF} \geq a-\frac{7}{2}$. This is true by lemma 2.6.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{TT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{F}\square \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq a \overset{\overset{3}{\rightarrow}}{-} \frac{7}{2}$.

Case 2.1: $\mathrm{T}^{a-2}\square\mathrm{T}\square\mathrm{FTF}\square\mathrm{F} \geq a-\frac{7}{2}$. This is II) case1 of lemma 2.6.

Case 2.2: $\mathrm{T}^{a-2}\square\mathrm{T}\square\mathrm{T} \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{FF}\square \geq a \overset{\overset{2}{\rightarrow}}{-} \frac{7}{2}$.

Case 2.2.1: $\mathrm{T}^{a-3}\square\mathrm{TTFT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{F}\square \geq a \overset{\overset{2}{\leftarrow}}{-} \frac{7}{2}$.

Case 2.2.1.1: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TF}\square \geq a \overset{\overset{2}{\leftarrow}}{-} \frac{7}{2}$.

Case 2.2.1.1.1: $\mathrm{T}^{a-3}\square\mathrm{TTFF}\square\mathrm{TF}\square \geq a-3$.

The left hand side is $(a-3)+0 = a-3$.

Case 2.2.1.1.2: $\mathrm{T}^{a-3}\square\mathrm{T}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TFTF}\square \geq a-4$.

$\Rightarrow \mathrm{T}^{a-3}\square\square\mathrm{FTTFTF}\square = a-4.$

The left hand side $\geq \mathrm{T}^{a-3}\square\square\mathrm{F} + \mathrm{FTF}\square = (a-4)+0 = a-4.$

Case 2.2.1.2: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TFF}\square \geq a-4.$

The left hand side $\geq \mathrm{T}^{a-3}\square + \square\mathrm{TFF}\square = (a-3)+0 = a-3.$

Case 2.2.2: $\mathrm{T}^{a-2}\square\square\mathrm{TTFFF}\square \geq a-4$. The left hand side is $2(a-2)$.

Case 2.3: $\mathrm{T}^{a-3}\square\mathrm{TTT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{F}\square \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq \ a-4.$

Case 2.3.1: $\mathrm{T}^{a-3}\square\mathrm{TT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TF}\square \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq \ a-4.$

Case 2.3.1.1: $a-4 \geq a-4$, by case II 1.1 of lemma 2.6.

Case 2.3.1.2: $\mathrm{T}^{a-3}\square\mathrm{T}\square\mathrm{TFTFF}\square \geq a-4.$

The left hand side $\Rightarrow \mathrm{T}^{a-3}\square\mathrm{TF}\square = a-3.$

Case 2.3.2: $\mathrm{T}^{a-3}\square\mathrm{TT}\square\mathrm{TFFF}\square \geq a-4$. The left hand side $\geq a-3$.

Case 3: $\mathrm{T}^{a-2}\square\mathrm{TT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}} \square \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \mathrm{F} \geq \ a-4.$

Case 3.1: $a-4 \geq a-4$, by lemma 2.6.

Case 3.2: $\mathrm{T}^{a-3}\square\mathrm{TTT}\square\mathrm{FF}\square\mathrm{F} \geq a-4$. This is true by the case 2.3

Theorem 5.4.1. $\mathrm{T}^{a}\square\square\square\mathrm{FFF} = a - \frac{7}{2}$, $a \geq 5$.

Need to show :

I) $\mathrm{T}^a\square\square\square\mathrm{FFF} \leq a - \frac{7}{2}$.

II) $\mathrm{T}^a\square\square\square\mathrm{FFF} \geq a - \frac{7}{2}$.

I) To Show $\overset{\xrightarrow{1}}{\mathrm{T}^a}\ \square\square\square\mathrm{FFF} \leq\ \overset{\xleftarrow{2}}{a - \frac{7}{2}}$.

Case 1: $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{FF} \leq a - \frac{7}{2}$. This is true by lemma 2.7.

Case 2: $\mathrm{T}^a\square\square\mathrm{F}\square\mathrm{FF} \leq a - 3$.

$\Rightarrow a - 3 \leq a - 3$, by lemma 2.7.

II) To Show $\mathrm{T}^a\square\square\square\ \overset{\xleftarrow{1}}{\mathrm{F}}\ \mathrm{FF} \geq\ \overset{\xrightarrow{2}}{a - \frac{7}{2}}$.

Case 1: $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{FFF} \geq a - \frac{7}{2}$. This is true by lemma 2.7.

Case 2: $\mathrm{T}^{a-1}\square\mathrm{T}\square\square\ \overset{\xleftarrow{1}}{\mathrm{F}}\ \mathrm{FF} \geq\ \ a - 4$.

$\Rightarrow a - 4 \geq a - 4$, by lemma 2.7.

The main theorem is proved. $\square$

5.5 $\mathrm{T}^a\square\square\mathrm{F}^b$, $a > b \geq 2$

We show $\mathrm{T}^a\square\square\mathrm{F}^b = \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}$, $a > b \geq 2$.

The position and value above was the first conjecture of Jeff Erickson in [4]. The proof is not long but tricky. We will prove 11 lemmas before we get the main theorem.

For the case $a > b = 2$, the result is already in chapter 2 and chapter 3.

For the case $a > b \geq 3$,We will prove 12 lemmas that will lead to the theorem.

Note: $\mathrm{T}^a\square\square\mathrm{F} = a-1, \quad a \geq 1$. (this result will be used in the lemma).

Below is how the tree looks like at the beginning.

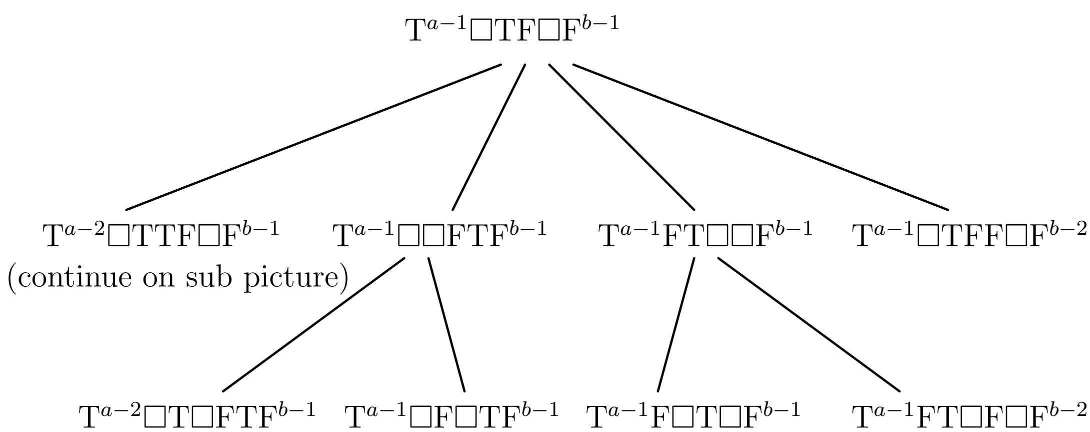


Figure 5.3: Main tree

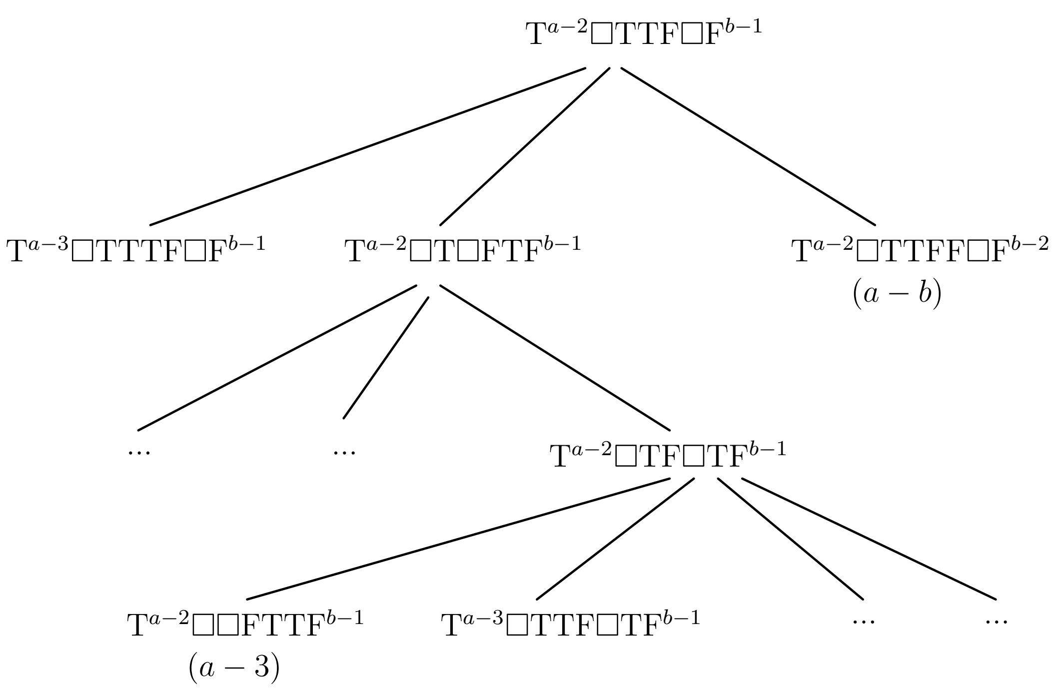


Figure 5.4: Sub picture of the main tree

Lemma 5.5.1. $\mathrm{T}^a\square\ \mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b = a, \quad k \geq 2, a \geq 0, b \geq 0.$

Proof by induction on a:

Base case: $a = 0$, $\square\ \mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b = 0$. The statement is true by Death Leap Principle.

The induction step:

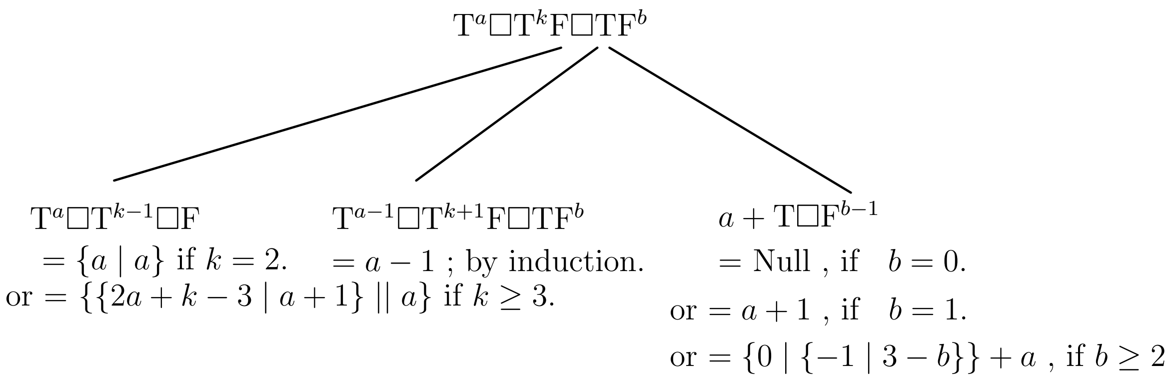


Figure 5.5: $\mathrm{T}^a\square\ \mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b = a$.

Lemma 5.5.2. $\mathrm{T}^a\square\mathrm{T}^k\square\mathrm{FTF}^b \geq a - 1, \quad k \geq 1, a \geq 1, b \geq 0.$

Proof by induction on a:

Base case: $a = 1, \mathrm{T}\square\mathrm{T}^k\square\mathrm{FTF}^b \geq 0.$

$\Rightarrow \square\mathrm{T}^{k+1}\mathrm{F}\square\mathrm{TF}^b \geq 0$. The statement is true by Death Leap Principle.

Induction step:

$$\mathrm{T}^a\square\mathrm{T}^k\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TF}^b \geq\ a \overset{\overset{2}{\rightarrow}}{-} 1\ .$$

Case 1: $a - 1 \geq a - 1$, by lemma 3.1.

Case 2: $T^{a-1}\Box T^{k+1}\Box FTF^{b} \geq a-2$, true by induction.

Lemma 5.5.3. $\overset{\overset{2}{\rightarrow}}{T^{a}}\, F\Box T^{i}F^{j}\, \overset{\overset{1}{\rightarrow}}{T}\, \Box F^{b} \leq \{0 \overset{\overset{3}{\leftarrow}}{|} 0\}, \quad i \geq 1, j \geq 1, a \geq 0, b \geq 1.$

Proof by induction on a: (I will omit the base case since it is the same as the induction step except no case 2).

Case1: $T^{a}F\Box T^{i}F^{j}\Box TF^{b} \leq 0$, true by one side DLP.

Case2: $T^{a-1}F\Box T^{i+1}F^{j}T\Box F^{b} \leq \{0 \mid 0\}$, true by induction.

Case3: $T^{a}F\Box T^{i}F^{j}TF\Box F^{b-1} \leq 0$, true by one side DLP.

Lemma 5.5.4. $T^{a}F\, \overset{\overset{2}{\rightarrow}}{T}\, \Box F^{k}\, \overset{\overset{1}{\rightarrow}}{T}\, \Box F^{b} \leq 0, \quad k \geq 1, a \geq 0, b \geq 2.$

Case1: $T^{a}F\, \overset{\overset{2}{\rightarrow}}{T}\, \Box F^{k+1}\, \overset{\overset{1}{\rightarrow}}{T}\, \Box F^{b-1} \leq 0$

Case1.1: $T^{a}FTF\Box F^{k}\Box TF^{b-1} \leq 0$, true by one side DLP.

Case1.2: $T^{a}F\Box TF^{k+1}TF\Box F^{b-2} \leq 0$, true by one side DLP.

Case2: $T^{a}F\Box TF^{k}TF\Box F^{b-1} \leq 0$, true by one side DLP.

Lemma 5.5.5. $T^{a}F\, \overset{\overset{1}{\rightarrow}}{T}\, \Box F^{k}\Box TF^{b} \leq \{0 \overset{\overset{2}{\leftarrow}}{|} 0\}, \quad k \geq 0, a \geq 0, b \geq 3.$

Case1: $T^{a}F\Box TF^{k+1}T\Box F^{b-1} \leq \{0 \mid 0\}$, true by lemma 3.3.

Case2: $T^{a}FT\Box F^{k+1}T\Box F^{b-1} \leq 0$, true by lemma 3.4.

Lemma 5.5.6. $\overset{\overset{2}{\rightarrow}}{T^{a}}\, \Box\, \overset{\overset{1}{\rightarrow}}{T}\, F\Box TF^{b} \leq T^{a}\Box TTF\Box\, \overset{\overset{3}{\leftarrow}}{F^{b}}, \quad a \geq b \geq 2.$

(For bypass reversible move in the future)

Case1: $\mathrm{T}^a\square\square\mathrm{F} \leq \mathrm{T}^a\square\mathrm{T}\square\mathrm{FTF}^b$

$\Rightarrow a-1 \leq \mathrm{T}^a\square\mathrm{T}\square\mathrm{FTF}^b$ since $\mathrm{T}^a\square\square\mathrm{F} = a-1$, by note. True by lemma 3.2.

Case2: $\mathrm{T}^{a-1}\square\mathrm{TTF}\square\mathrm{TF}^b \leq \mathrm{T}^a\square\mathrm{T}\square\mathrm{FTF}^b$.

The left hand side is $a-1$ by lemma 3.1.

Then the statement is true by lemma 3.2.

Case3: $\mathrm{T}^a\mathrm{FT}\square\square\mathrm{TF}^b \leq a-b+1$

$\Rightarrow \mathrm{T}^a\mathrm{F}\square\mathrm{TFT}\square\mathrm{F}^{b-1} \leq a-b+1$, true by lemma 3.3.

Lemma 5.5.7. $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^a}\,\square\mathrm{TT}\,\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{F}^b \leq a, \quad a \geq 0, b \geq 1.$

Case 1: $\mathrm{T}^a\square\mathrm{TTF}\square\mathrm{TF}^b \leq a$, true by lemma 3.1.

Case 2: $\mathrm{T}^{a-1}\square\mathrm{TTTTFF}\square\mathrm{F}^{b-1} \leq a$. The left hand side is $a-b$.

Lemma 5.5.8. $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\,\square\,\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{TF}^b \leq \mathrm{T}^a\square\square\,\overset{\overset{3}{\leftarrow}}{\mathrm{F}}\,\mathrm{TF}^b, \quad a \geq 2, b \geq 0.$

(For bypass reversible move in the future)

Case 1: $a-2 \leq \mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{FTF}^b$, true by lemma 3.2.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{TTF}\square\mathrm{TF}^b \leq \mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{FTF}^b$, (the left hand side $= a-2$, by lemma 3.1), true by lemma 3.2.

Case 3: $\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{TF}^b \leq \mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{TF}^b$.

Lemma 5.5.9. $\mathrm{T}^a\square\mathrm{F}\square\mathrm{TF}^b \leq 1, \quad a \geq 2, b \geq 2.$

$\Rightarrow \mathrm{T}^{a-1}\mathrm{FT}\square\square\mathrm{TF}^{b} \leq 1.$

$\Rightarrow \mathrm{T}^{a-1}\mathrm{F}\square\mathrm{TFT}\square\mathrm{F}^{b-1} \leq 1$, true by lemma 3.3.

Lemma 5.5.10. $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\square\mathrm{F}^{b} = \{0 \mid 0\}, \quad k \geq 1, a \geq 3, b \geq 2.$

Need to show:

I) $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\square\mathrm{F}^{b} \leq \{0 \mid 0\}, \quad k \geq 1, a \geq 0, b \geq 2.$

II) $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\square\mathrm{F}^{b} \geq \{0 \mid 0\}, \quad k \geq 1, a \geq 3, b \geq 1.$

I) To show $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a}}\ \mathrm{F}\square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}^{k}}\ \square\mathrm{F}^{b} \leq\ \{0 \overset{\overset{3}{\leftarrow}}{\mid} 0\}, \quad k \geq 1, a \geq 0, b \geq 2.$

Prove by induction on a: (I will omit the base case since it is the same as the induction step except no case 2).

Induction step:

Case 1: $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k-1}\mathrm{FT}\square\mathrm{F}^{b-1} \leq \{0 \mid 0\}$, true by lemma 3.3.

Case 2: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}^{k+1}\square\mathrm{F}^{b} \leq \{0 \mid 0\}$, true by induction.

Case 3: $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\mathrm{F}\square\mathrm{F}^{b-1} \leq 0$, true by one side DLP.

II) To show $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\square\ \overset{\overset{1}{\leftarrow}}{\mathrm{F}^{b}} \geq \{0 \overset{\overset{2}{\rightarrow}}{\mid} 0\}, \quad k \geq 1, a \geq 3, b \geq 1.$

Case 1: $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k-1}\square\mathrm{FTF}^{b-1} \geq \{0 \mid 0\}$, true by negative of lemma 3.5.

Case 2: $\mathrm{T}^{a-1}\Box \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \mathrm{T}^{k+1}\Box\ \overset{\overset{1}{\leftarrow}}{\mathrm{F}}{}^{b} \geq 0$.

Case 2.1: $\mathrm{T}^{a-1}\mathrm{F}\Box\mathrm{T}^{k}\Box\mathrm{T}\mathrm{F}^{b} \geq 0$, true by one side DLP.

Case 2.2: $\mathrm{T}^{a-1}\Box\mathrm{F}\mathrm{T}^{k}\Box\mathrm{F}\mathrm{T}\mathrm{F}^{b-1} \geq 0$, true by negative of lemma 3.4.

Lemma 5.5.11. $\{* \mid \overset{\overset{1}{\rightarrow}}{1} - b\} \ \leq \mathrm{T}^{a}\Box\mathrm{T}\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \mathrm{F}\Box\ \overset{\overset{3}{\leftarrow}}{\mathrm{F}}{}^{b}, \quad b \geq 1, a \geq b+2.$

Case 1: $* \leq a - b - 1$.

Case 2: $\{* \mid \overset{\overset{1}{\rightarrow}}{1} - b\} \leq \mathrm{T}^{a}\mathrm{F}\Box\mathrm{T}\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \Box\ \overset{\overset{3}{\leftarrow}}{\mathrm{F}}{}^{b}$.

Case 2.1: $* \leq \mathrm{T}^{a}\mathrm{F}\Box\Box\mathrm{F}\mathrm{T}\mathrm{F}^{b}$, true by the negative of lemma 3.5.

Case 2.2: $1 - b \leq \mathrm{T}\Box\Box\mathrm{F}^{b}$. The right hand side is $1 - b$, by note.

Case 2.3: $1 - b \leq \mathrm{T}^{a}\mathrm{F}\Box\mathrm{T}\mathrm{F}\mathrm{F}\Box\mathrm{F}^{b-1}$, true by the negative of lemma 3.1.

Case 3: $\{* \mid 1 - b\} \leq a - b$.

$\Rightarrow 0 \leq a - b$, which is true.

After applying lemma 1,6,7,8,9,10,11 to the tree, it looks like:

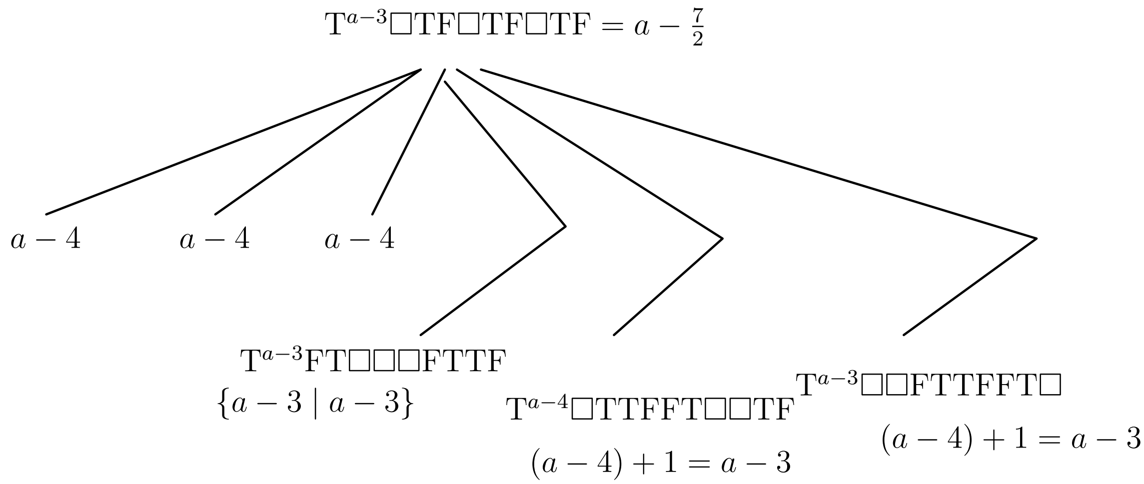


Figure 5.6: Tree after applying lemma 1,6,7,8,9,10,11

After applying these lemmas, we have

$\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{F}^{b-1} = \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}.$

We finally the main theorem using the result above.

Theorem 5.5.1. $\mathrm{T}^{a}\square\square\mathrm{F}^{b} = \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}, \;\; a > b \geq 3.$

Need to show:

I) $\mathrm{T}^{a}\square\square\mathrm{F}^{b} \geq \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}, \;\; a > b \geq 3.$

II) $\mathrm{T}^{a}\square\square\mathrm{F}^{b} \leq \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}, \;\; a > b \geq 3.$

I) To show $\mathrm{T}^{a}\square\square \overset{\xleftarrow{1}}{\mathrm{F}}{}^{b} \;\geq\; \{\{a-3 \mid a-\overset{\xrightarrow{2}}{b}\} \mid \{* \mid 3-b\}\}, \;\; a > b \geq 3.$

Case 1: $\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{F}^{b-1} \geq \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}$, true by the tree above.

Case 2: $\mathrm{T}^{a-1}\square\mathrm{T}\square \overset{\xleftarrow{1}}{\mathrm{F}}{}^{b} \;\geq\; \{a-3 \overset{\xrightarrow{2}}{\mid} a-b\}.$

Case 2.1: $\{a-3 \mid a-b\} \geq \{a-3 \mid a-b\}$, by the tree above.

Case 2.2: $\mathrm{T}^{a-2}\square\mathrm{TT}\square\mathrm{F}^{b} \geq a-3$.

We will prove case 2.2 in the more general cases.

$\mathrm{T}^{a-2}\square\mathrm{T}^{k}\square\mathrm{F}^{b} \geq a-3, \quad k \geq 2, a \geq 3,$ by induction on a.

Base Case: $a=3$, **To show** $\mathrm{T}\square\mathrm{T}^{k}\square\mathrm{F}^{b} \geq 0$.

$\Rightarrow \mathrm{T}\square\mathrm{T}^{k-1}\square\mathrm{FTF}^{b-1} \geq 0$, true by lemma 3.2.

Induction step: $\mathrm{T}^{a-2}\square\mathrm{T}^{k}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}^{b}} \geq a \overset{\overset{2}{\rightarrow}}{-} 3$.

Case1: $\mathrm{T}^{a-2}\square\mathrm{T}^{k-1}\square\mathrm{FTF}^{b-1} \geq a-3$, true by lemma 3.2.

Case2: $\mathrm{T}^{a-3}\square\mathrm{T}^{k+1}\square\mathrm{F}^{b} \geq a-4$, true by induction.

II) To show $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a}} \square\square\mathrm{F}^{b} \leq \{\{a-3 \mid a-\overset{\overset{2}{\leftarrow}}{b}\} \mid \{* \mid 3-b\}\}, \quad a > b \geq 3$.

Case 1: $\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{F}^{b-1} \leq \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}$, true by the tree above.

Case 2: $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a}} \square\mathrm{F}\square\mathrm{F}^{b-1} \leq \{* \mid 3 \overset{\overset{2}{\leftarrow}}{-} b\}$.

Case 2.1: $\{* \mid 3-b\} \leq \{* \mid 3-b\}$, by lemma 3.11.

Case 2.2: $\mathrm{T}^{a}\square\mathrm{FF}\square\mathrm{F}^{b-2} \leq 3-b$. This is the negative of case **I)** 2.2 above.

The theorem is proved. $\square$

5.6 $\mathrm{T}^a\square\square\square\mathrm{F}^b, \quad a \geq 4, \quad b \geq 4$

We show $\mathrm{T}^a\square\square\square\mathrm{F}^b = \{a-b \mid a-b\}, \quad a \geq 4, \quad b \geq 4.$

The values of the starting positions are of a great interest. Some of them have been investigated in Erickson's paper [4]. The starting position with the variables on both Toads and Frogs are interested the author a lot. We showed the value of $\mathrm{T}^a\square\square\mathrm{F}^b, a > b \geq 2$ in the previous section. In this section we show the value of $\mathrm{T}^a\square\square\square\mathrm{T}^b, a \geq 4, b \geq 4$. It is still an open problem about the values of the position $\mathrm{T}^a\square\square\square\square\mathrm{T}^b, a \geq 6, b \geq 6$ and $\mathrm{T}^a\square\square\square\square\square\mathrm{T}^b$.

We will do the case analysis similar to the one in the previous two sections. The proof here is not long. The outline is below.

$$\mathrm{T}^a\square\square\square\mathrm{F}^b = \{a-b \mid a-b\}.$$

$$\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{F}^{b-1} = \{a-b \mid a-b\}.$$

$$\mathrm{T}^{a-2}\square\mathrm{TT}\square\mathrm{FF}\square\mathrm{F}^{b-2} = \{a-b \mid a-b\}.$$

$$\mathrm{T}^{a-3}\square\mathrm{TTT}\square\mathrm{FFF}\square\mathrm{F}^{b-3} = \{a-b \mid a-b\}.$$

Figure 5.7: $\mathrm{T}^a\square\square\square\mathrm{F}^b = \{a-b \mid a-b\}$

Lemma 5.6.1. $\mathrm{T}^{a-3}\square\mathrm{T}^3\square\mathrm{F}^3\square\mathrm{F}^{b-3} = \{a-b \mid a-b\}, \quad a \geq 4, b \geq 4.$

By symmetry, we only need to show $\overset{\xrightarrow{2}}{\mathrm{T}^{a-3}} \square \overset{\xrightarrow{1}}{\mathrm{T}^3} \square\mathrm{F}^3\square\mathrm{F}^{b-3} \leq \{a-b \overset{\xleftarrow{3}}{\mid} a-b\}$.

Case 1: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-3}} \square\mathrm{TTF}\overset{\xrightarrow{1}}{\mathrm{T}} \square\mathrm{FF}\square\mathrm{F}^{b-3} \leq \{a-b \overset{\xleftarrow{3}}{\mid} a-b\}$.

Case 1.1: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TFF}\square\mathrm{F}^{b-3} \leq a-b$, true by one side DLP.

Case 1.2: $\mathrm{T}^{a-4}\square\mathrm{TTTFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$, true by one side DLP.

Case 1.3: $\mathrm{T}^{a-3}\square\mathrm{TTFTF}\square\mathrm{F}\square\mathrm{F}^{b-3} \leq a-b$.

$\Rightarrow \mathrm{T}^{a-3}\square\mathrm{TTFF}\square\mathrm{TF}\square\mathrm{F}^{b-3} \leq a-b$.

The left hand side is $(a-3)-(b-3) = a-b$.

Case 2: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-4}} \square\mathrm{TTT}\overset{\xrightarrow{1}}{\mathrm{T}} \square\mathrm{FFF}\square\mathrm{F}^{b-3} \leq a-b$.

Case 2.1: $\mathrm{T}^{a-4}\square\mathrm{TTTFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$. This is case 1.2.

Case 2.2: $\mathrm{T}^{a-5}\square\mathrm{TTTTTF}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$.

The left hand side is $\leq (a-5)-(b-3) = a-b-2$.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{TTTF}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$. The left hand side is $\leq a-b$.

Lemma 5.6.2. $\mathrm{T}^{a-2}\square\mathrm{T}^2\square\mathrm{F}^2\square\mathrm{F}^{b-2} = \{a-b \mid a-b\}, \quad a \geq 4, \quad b \geq 4.$

By symmetry, we only need To show $\overset{\xrightarrow{2}}{\mathrm{T}^{a-2}} \square \overset{\xrightarrow{1}}{\mathrm{T}^2} \square\mathrm{F}^2\square\mathrm{F}^{b-2} \leq \{a-b \overset{\xleftarrow{3}}{\mid} a-b\}$.

Case 1: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-2}} \square\mathrm{TF}\overset{\xrightarrow{1}}{\mathrm{T}} \square\mathrm{F}\square\mathrm{F}^{b-2} \leq \{a-b \overset{\xleftarrow{3}}{\mid} a-b\}$.

Case 1.1: $\mathrm{T}^{a-2}\square\mathrm{TFFT}\square\square\mathrm{F}^{b-2} \leq \{a-b \mid a-b\}$.

Left has to move the left most T. Since otherwise right will move the left most F and block the left most position. So we have

$\Rightarrow \mathrm{T}^{a-3}\square\mathrm{TTFFT}\square\square\mathrm{F}^{b-2} \leq a-b$. The left hand side is $(a-3)-(b-3) = a-b$.

Case 1.2: $\mathrm{T}^{a-3}\square\mathrm{TTFT}\square\mathrm{FF}\square\mathrm{F}^{b-2} \leq \{a-b \mid a-b\}$. This is case 1 in lemma 4.1.

Case 1.3: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-2}}\ \square\mathrm{TF}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$.

Case 1.3.1: $\mathrm{T}^{a-2}\square\mathrm{TFFT}\square\mathrm{F}\square\mathrm{F}^{b-3} \leq a-b$.

Left has to move the left most T. Since otherwise right will move the left most F and block the left most position. So we have

$\Rightarrow \mathrm{T}^{a-3}\square\mathrm{TTFFTF}\square\square\mathrm{F}^{b-3} \leq a-b$. The left hand side is $(a-3)-(b-3) = a-b$.

Case 1.3.2: $\mathrm{T}^{a-3}\square\mathrm{TTFTF}\square\mathrm{F}\square\mathrm{F}^{b-3} \leq a-b$.

$\Rightarrow \mathrm{T}^{a-3}\square\mathrm{TTFF}\square\mathrm{TF}\square\mathrm{F}^{b-3} \leq a-b$.

The left hand side is $(a-3)-(b-3) = a-b$.

Case 2: $\mathrm{T}^{a-3}\square\mathrm{TTT}\square\mathrm{FFF}\square\mathrm{F}^{b-3} \leq \{a-b \mid a-b\}$, true by lemma 4.1.

Case 3: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-2}}\ \square\mathrm{T}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \square\mathrm{FFF}\square\mathrm{F}^{b-3} \leq a-b$.

Case 3.1: $\mathrm{T}^{a-2}\square\mathrm{TFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$. This is case 1.3 above.

Case 3.2: $a-b \leq a-b$,   by lemma 4.1.

Lemma 5.6.3. $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{F}^{b-1} = \{a-b \mid a-b\}, \quad a \geq 4, \quad b \geq 4.$

By symmetry, we only need To show $\overset{\xrightarrow{2}}{\mathrm{T}^{a-1}} \square \overset{\xrightarrow{1}}{\mathrm{T}} \square\mathrm{F}\square\mathrm{F}^{b-1} \leq \{a-\overset{\xleftarrow{3}}{b \mid} a-b\}.$

Case 1: $\overset{\xrightarrow{1}}{\mathrm{T}^{a-1}} \square\square\mathrm{TFF}\square\mathrm{F}^{b-2} \leq \{a-\overset{\xleftarrow{2}}{b \mid} a-b\}.$

Case 1.1: $\mathrm{T}^{a-2}\square\mathrm{TFT}\square\mathrm{F}\square\mathrm{F}^{b-2} \leq \{a-b \mid a-b\}.$ This case 1 of lemma 4.2.

Case 1.2: $\mathrm{T}^{a-1}\square\square\mathrm{TFFF}\square\mathrm{F}^{b-3} \leq a-b.$

$\Rightarrow \mathrm{T}^{a-2}\square\mathrm{TFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$. This is case 1.3 of lemma 4.2.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{TT}\square\mathrm{FF}\square\mathrm{F}^{b-2} \leq \{a-b \mid a-b\}$, true by lemma 4.2.

Case 3: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-1}} \square \overset{\xrightarrow{1}}{\mathrm{T}} \square\mathrm{FF}\square\mathrm{F}^{b-2} \leq a-b.$

Case 3.1: $\mathrm{T}^{a-1}\square\square\mathrm{TFFF}\square\mathrm{F}^{b-3} \leq a-b.$ This is case 1.2 above.

Case 3.2: $a-b \leq a-b$, by lemma 4.2.

Theorem 5.6.1. $\mathrm{T}^{a}\square\square\square\mathrm{F}^{b} = \{a-b \mid a-b\}, \quad a \geq 4, \quad b \geq 4.$

By symmetry, we only need To show $\overset{\xrightarrow{1}}{\mathrm{T}^{a}} \square\square\square\mathrm{F}^{b} \leq \{a-\overset{\xleftarrow{2}}{b \mid} a-b\}.$

Case 1: $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{F}^{b-1} \leq \{a-b \mid a-b\}$, true by lemma 4.3.

Case 2: $\mathrm{T}^{a}\square\square\mathrm{F}\square\mathrm{F}^{b-1} \leq a-b.$

$\Rightarrow a-b \leq a-b$, by lemma 4.3.

The theorem is proved. □

Chapter 6

How to beat Capablanca

6.1 Introduction

The first thing Capablanca mentions in his book, *Chess Fundamentals*, is how to checkmate with rook, as in Figure 6.1.

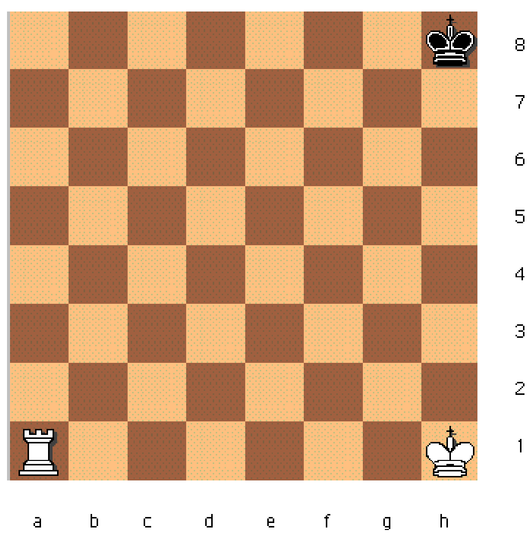


Figure 6.1: Starting position.

Capablanca writes,

> In this position the power of the rook is demonstrated by the first move, Ra7, which immediately confines the black king to the last rank, ...

Capablanca did not give the fastest way to checkmate. With **1.Ra7**, the fastest way for White to checkmate Black king is 10 moves. While with **1.Rg1**, White can force a checkmate in 9 moves.

6.2 On an $m \times n$ board.

For the general $m \times n$ board with White king (WK) on $(m, 1)$, White rook (WR) on (1,1), and Black king (BK) on (m, n) where $m \geq 4$ and $n \geq 5$, we define $FM(m, n)$ to be the smallest number of moves for white to checkmate Black king.

Theorem. $FM(m, n) = \begin{cases} n & \textit{if } n \textit{ is odd;} \\ n+1 & \textit{if } n \textit{ is even.} \end{cases}$

Let's call the right side $G(m, n)$. We will prove this by

1) Showing a sequence of White moves that force the checkmate in $G(m, n)$ moves.

2) Showing that Black has a strategy to survive up to $G(m, n) - 1$ moves.

Lemma 1. *In the given postion on an $m \times n$ board, White can give a checkmate in n moves if n is odd.*

I will give the sequence of white moves so that for all the choices of Black moves, white can give the checkmate in n moves.

First move: 1.WR(m-1,1) BK(m,n-1) (only move)

Then White can make a sequence of White king moves up one square no matter what Black responds are. Black's only responds are moving the king up and down along the m^{th}column. But this can not interrupt White king because of the parity. It will take $n - 4$ moves for white king to move up to the square $(m, n - 3)$. We have Figure 6.2 below with White to move.

White can finish this off by playing the forcing move

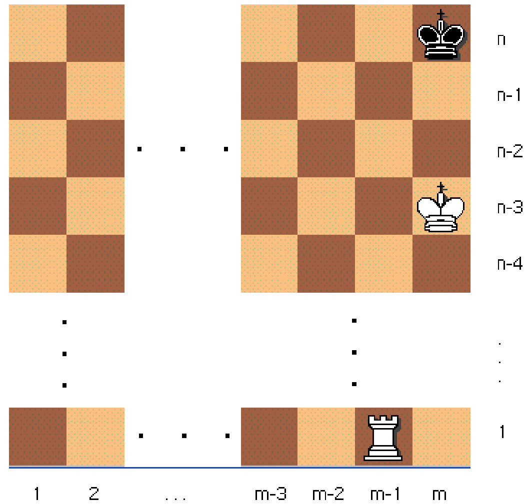


Figure 6.2: Position before checkmate.

1. WK(m-1,n-2) BK(m-1,n) (only move)
2. WR(m-2,1) BK(m,n) (only move)
3. WR(m-2,n) Checkmate

The total number of moves is $1 + (n - 4) + 3 = n$.

Lemma 2. *White can force a checkmate in $n + 1$ moves, if n is even.*

The strategy is almost the same as in Lemma 1. However, because of the parity we have to make one waiting move, i.e.,WR(m-1,2) while White king tries to move upward. Since $n \geq 5$, we have enough room for this plan.

Note: The first move WR(m-1,1) with the strategy above gives White the fastest way to mate. As we can see it requires at least $n - 3$ (resp. $n - 2$ moves) for n odd (resp. even) for White King to move up the board and at least 2 more Rook moves before White can force a checkmate.

Lemma 3. *Black has a strategy to survive up to $n - 1$ moves, if n is odd.*

The general plan for Black to survive is to move to the middle of the board as much as possible. At some point White has to move his Rook to restrict the possible moves of Black king. Then he must use White king to push Black king to the edge of the board. In addition, the sooner the Rook moves, the better. Therefore we will assume the first move is a Rook move. Note that the only mating positions are when Black king is at the edge of the board. Now we consider 2 cases of the first Rook move:

Case 1 The first rook move is vertical (moving along a column).

In this case, Black tries to move to the middle of the board as much as possible. Once Black king gets there, he will try to stay there as long as possible before he is forced to the corner. Below is an example where the size of the board is 9×9.

1. WR(1,8)	BK(8,9)
2. WK(8,2)	BK(7,9)
3. WK(7,3)	BK(6,9)
4. WK(6,4)	BK(5,9)
5. WK(5,5)	BK(6,9)
6. WK(5,6)	BK(5,9)
7. WK(4,7)	BK(6,9)
8. WK(5,7)	BK(7,9)
9. WK(6,7)	BK(8,9)
10.WK(7,7)	BK(9,9)
11.WK(8,7)	BK(8,9)
12.WR(1,9)	Checkmate

On $m \times n$ board, we can see that White has to move his Rook twice, and White king moves up the board in $n - 3$ moves and chases Black king back to the corner with at least another $\lceil(\frac{m}{2})\rceil - 1$ moves. For example on a 9×9 board, it takes $2 + (9 - 3) + (5 - 1) = 12$ moves. This is the best that White can do. Therefore

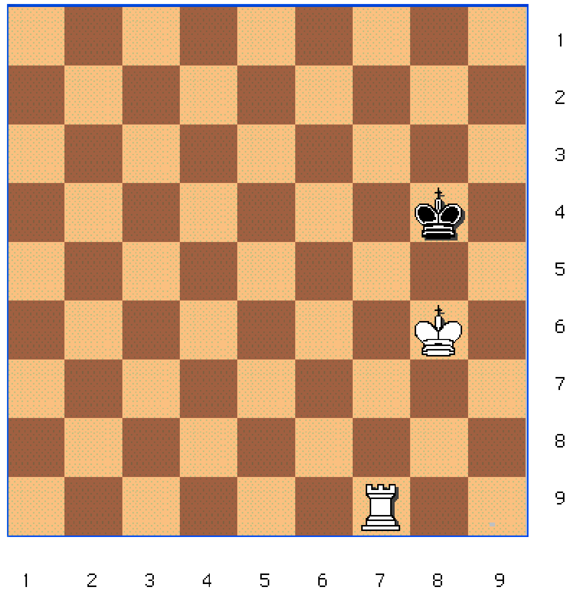


Figure 6.3: Intermediate position.

$F(m,n) = 2 + (n-3) + (\lceil(\frac{m}{2})\rceil - 1) \geq n$ moves (since $m \geq 4$).

Case 2 The first Rook move is horizontal (moving along a row).

(We exclude the move $R(m-1,1)$ since we already know that in that case the fastest number of moves to mate is n.)

Black king will try to move down and toward the middle of the board as much as possible. Once he gets blocked by the rook and the White king, Black king moves along the row (otherwise Black King will just try to stay in the middle of the board). He could move along the row since the Rook is not at column $m-1$. Figure 3 illustrates the situation:

1. ... BK(9,6) Now white has two choices: 2.WK(7,5) or 2.WR(7,6). For 2.WK(7,5), Black's response will be BK(8,7) and the best white response is 3.WK(8,5). In this case, White King needs one extra move to get to the row $n-3$.

Overall, White has to move the rook twice (the first rook move and the checkmate move); the king takes $n-3$ moves to go up the board and at least an extra move from king or rook of the two choices above. This makes the least possible number of moves to checkmate at least $\geq 2+(n-3)+1=n$.

Lemma 4. *Black has a strategy to survive up to n moves, when n is even.*

The general plan in this case is similar. The strategy in Case 1 also works here. In Case 2, Black now has an advantage of parity. This will give him an extra move to survive when he faces White king in the middle of the board. The details of the proof are left as an exercise to the reader.

By Lemmas 1, 2, 3 and 4, we conclude the theorem.
The induction method using to solve the above position could be later on generalized to more complicated starting postion with different pieces on the board.

6.3 About José Raul Capablanca

Jose Raul Capablanca was born in Havana, Cuba on November 19, 1888. He is regarded as one of the most gifted chess players of all time. According to Capablanca, at the age of four, he learned the rules of chess by watching his father play chess with a friend. In 1921, he won the world champion title from mathematician and chess player Emanuel Lasker, who held the title for 27 years. Capablanca was the third official world chess champion. He lost his title to Alexander Alekhine in 1927 and died in 1942.

The move that Capablanca suggested is Case 1 of Lemma 4. Black could survive at least $2+(n-3)+(\lceil(\frac{m}{2})\rceil-1)=2+5+3=10$.

Chapter 7

On the Monochromatic Schur Triples problem

7.1 Introduction

The *Schur number*, $s(r)$, denotes the maximal integer n such that there exists an r-coloring of $[1, n-1]$ that avoids a monochromatic solution to $x + y = z$. For example $s(2) = 5$ and $s(3) = 14$. $s(5)$ is unknown but is conjectured to be 161.

The original question about the minimum number, over all 2-colorings of $[1, n]$, of monochromatic Schur triples was asked by Ronald Graham in 1997. It can be thought of as a larger-scale version of the problem asked by the Schur numbers. It was solved in 1998. The answer is $\frac{n^2}{22} + O(n)$, that is realized by coloring the first $\frac{4n}{11}$ integers red, the next $\frac{6n}{11}$ integers blue, and the final $\frac{n}{11}$ integers red. The first two solutions were given by Robertson and Zeilberger [9] and Schoen [10]. Later Datskovsky [3] found another proof.

Ronald Graham asked another question generalizing the original one. The question was about the minimum number of monochromatic $(x, y, x+ay)$ triples, $a \geq 2$ on $[1, n]$. We discuss this problem in this chapter.

In section 2, we give a new simple proof of the original problem of finding the minimum number, over all 2-colorings of $[1, n]$, of monochromatic Schur triples. In section 3, we talk about the generalized problem asked by Graham. For this problem, we wrote a computer program to find an optimal coloring for small n to see some patterns. T hen we used a newly found "greedy calculus" to obtain a "good" upper bound. The final step was to try to match the lower bound and upper bound of the problem. In section 4, we also apply the greedy calculus to the original question on Schur triples with $r \geq 3$, to obtain a new upper bound.

7.2 The minimum number, over all 2-colorings of $[1, n]$, of monochromatic Schur triples

7.2.1 A Greedy Algorithm for The Upper bound

It is natural to find examples of good colorings first. This example will give us an upper bound. Then we try to show that this upper bound is also a lower bound.

We will show how to find an upper bound for the minimum number, over all 2-colorings of $[1, n]$, of monochromatic triples that are solutions of $x + y = z$. We will obtain this upper bound by using the Greedy Algorithm. We denote the colors red and blue.

The general idea is to keep adding more new intervals with different colors so that, each time, the overall coloring has the least number of monochromatic triples. For other proofs of this original problem see [9], [3], [10].

First

We paint the first interval of length k red. We will have $\frac{k^2}{4} + O(k)$ monochromatic triples solution of $x + y = z$.

Second

We paint the second interval blue. We want to find the length of the interval (with this color) so that the overall number of the monochromatic triples is minimized.

Let the length of this interval be $(1+j)k$ (here j is the number we want to find).

The total number of monochromatic triples on the whole interval is now $\frac{k^2}{4}+\frac{j^2k^2}{4}=\frac{(1+j^2)k^2}{4}$.

The total length is N $= k+(1+j)k=(2+j)k$.

So the total number of monochromatic triples in terms of N is

$\frac{(1+j^2)(\frac{N}{2+j})^2}{4}=\frac{(1+j^2)}{(2+j)^2}\frac{N^2}{4}$.

To find the minimum, we use calculus to get $j=\frac{1}{2}$. The total number of monochromatic Schur triples is then $\frac{N^2}{20}+O(n)$.

So far so good. We have a coloring that paints the first k integers red, followed by painting the next $(1+\frac{1}{2})k$ integers blue.

Third

Now we try to stick red at the end of the interval, and try to lower the overall number of triples. Say the length of this interval is jk, where j is the number we want to find. The total number of monochromatic Schur triples on the whole interval is $\frac{k^2}{4}+\frac{k^2}{16}+\frac{j^2k^2}{2}=(\frac{5}{16}+\frac{j^2}{2})k^2$.

The total length is $N=k+(1+\frac{1}{2})k+jk=(\frac{5}{2}+j)k$.

So the total number of monochromatic Schur triples in term of N is

$(\frac{5}{16}+\frac{j^2}{2})\frac{N^2}{(\frac{5}{2}+j)^2}=\frac{5+8j^2}{(5+2j)^2}\frac{N^2}{4}$.

To find the minimum, we again use calculus and get $j=\frac{1}{4}$. The total number of monochromatic triples is $\frac{N^2}{22}+O(n)$. The coloring for the whole interval is a red interval of length equal to k, a blue interval of length equal to $(1+\frac{1}{2})k$ and another red interval of length equal to $\frac{1}{4}k$. k is such that the sum of these intervals is N, i.e. $k=\frac{N}{(\frac{5}{2}+\frac{1}{4})}=\frac{4N}{11}$.

<u>**Fourth**</u>

We try to lower the bound further by having a blue interval of length, say, jk at the end of the previous interval. But now we get that the minimizing j is negative. So we stop.

As a conclusion, the optimal coloring is proportional to $[1, \frac{3}{2}, \frac{1}{4}]$, with colors $[R, B, R]$ yielding that indeed the minimal number is $\frac{N^2}{22} + O(n)$.

7.2.2 The Lower Bound

Finding lower bound is, in general, the difficult part. However, in this case, it is possible since we can turn the problem into a calculus problem. A similar technique was used in [8]

Definition
Let $M_\chi(n)$ be the number of monochromatic Schur triples for a 2-coloring χ of $[1, n]$.
Let Q be two times the number of non-monochromatic Schur triples for a 2-coloring of $[1, n]$.

Divide the interval $[1, n]$ into k consecutive intervals.
Let r_i be the number of red points in the interval I_i.
Let b_i be the number of blue points in the interval I_i.
Let $S_{i,j}$ be the number of non-monochromatic pairs in the square of $I_i \times I_j$.
Let $T_{i,j}$ be the number of non-monochromatic pairs in the triangle of $I_i \times I_j$.

Note: $r_i + b_i = \frac{n}{k}$.

Lemma 1) $M_\chi(n) = \frac{n^2}{4} - \frac{Q}{2} + O(n)$.

The total number of triples = |monochromatic triples| + |non-monochromatic triples|

$$= M_\chi(n) + \frac{Q}{2}\ .$$

Since the total number of triples is $\frac{n^2}{4} + O(n)$, we have $M_\chi(n) = \frac{n^2}{4} - \frac{Q}{2} + O(n)$. □

The plan is to find an upper bound of Q that will give the lower bound of $M_\chi(n)$.

Lemma 2) $Q = |R|\,|B| + \frac{1}{2}(\sum_{i+j<k} S_{i,j} + \sum_{i=1..k} T_{i,k-i+1})$,
where $|R| = \sum_{i=1..k} r_i$ and $|B| = \sum_{i=1..k} b_i$.

$$\begin{aligned} Q &= |\{(R,B),(B,R)|\ \ y-x \geq 0\}| + |\{(R,B),(B,R)|\ \ x+y \leq n, x \geq y\}|\,. \\ &= |\{(R,B),(B,R)|\ \ y-x \geq 0\}| + \tfrac{1}{2}\,|\{(R,B),(B,R)|\ \ x+y \leq n\}|\,. \end{aligned}$$

Note that each non-monochromatic triple contributes two non-monochromatic pairs: for example, $(x,y,z) = (R,B,R)$ gives $(x,y) = (R,B)$ and $(y,z) = (B,R)$. The statement of the lemma follows. □.

Now we find an upper bound for Q.

For each $T_{i,j}$ we have two ways to bound it:
1) $T_{i,j} \leq$ area of the triangle $= \frac{1}{2}(\frac{n}{k})^2$.
2) $T_{i,j} \leq S_{i,j}$.

Example 1: $k = 2$, with the upper bound of $T_{1,2}, T_{2,1}$ using the areas of the triangles. We have

$$\begin{aligned} Q &= |R|\,|B| + \tfrac{1}{2}(S_{1,1} + T_{1,2} + T_{2,1}). \\ &\leq (r_1 + r_2)(b_1 + b_2) + r_1 b_1 + \tfrac{n^2}{8}. \\ &= (r_1 + r_2)(n - r_1 - r_2) + r_1(\tfrac{n}{2} - r_1) + \tfrac{n^2}{8}. \end{aligned}$$

We use calculus to find a maximum of Q where $0 \leq r_1, r_2 \leq \frac{n}{2}$. The optimal solutions is $r_1 = \frac{n}{4}$ and $r_2 = \frac{n}{4}$.

We then get the maximum Q as $\frac{7n^2}{16}$. This yields $M_\chi(n) \geq \frac{n^2}{32} + O(n)$.$\square$

Example 2: $k = 3$, with the upper bound of $T_{1,3}, T_{3,1}$ using the areas of the triangles. The upper bound of $T_{2,2}$ using $S_{2,2}$. We have

$$\begin{aligned} Q &= |R|\,|B| + \tfrac{1}{2}(S_{1,1} + S_{1,2} + S_{2,1} + T_{1,3} + T_{2,2} + T_{3,1}). \\ &\leq (r_1 + r_2 + r_3)(b_1 + b_2 + b_3) + r_1 b_1 + r_1 b_2 + r_2 b_1 + r_2 b_2 + \tfrac{n^2}{18}. \end{aligned}$$

We use calculus to find a maximum of Q where $0 \leq r_1, r_2, r_3 \leq \frac{n}{3}$. One of the optimal solution is $r_1 = 0, r_2 = \frac{n}{3}$ and $r_3 = \frac{n}{6}$.

This yields the maximum Q is $\frac{5n^2}{12}$ which leads to $M_\chi(n) \geq \frac{n^2}{24} + O(n)$.$\square$

This is pretty nice. We can use calculus to get a decent lower bound of the problem. The calculation can even be done by hand. The hope to match the upper bound and lower bound is to try 11 intervals. This time we need a computer to help doing the calculation.

Example 3: $k = 11$,

We bound $T_{2,10}, T_{3,9}, T_{4,8}, T_{8,4}, T_{9,3}$ and $T_{10,2}$ by the area of each triangle which is $\frac{n^2}{242}$.

We bound $T_{i,12-i}$ by $S_{i,12-i}$,where $i = 1, 5, 6, 7, 11$.

We get eight optimal solutions to the maximum of Q. One of them is

$[r_1, r_2, ..., r_{11}] = [\frac{n}{11}, \frac{n}{11}, \frac{n}{11}, \frac{n}{11}, 0, 0, 0, 0, 0, 0, \frac{n}{11}]$.

This yields the maximum of Q as $\frac{9n^2}{22}$ which gives $M_\chi(n) \geq \frac{n^2}{22} + O(n)$. $\square$

YAY!! Since the lower bound matches the upper bound, the problem is solved.

7.3 Generalized problem, $x + ay = z$, $a \geq 2$

7.3.1 A Greedy Algorithm for Upper bounds

We will show how to find an upper bound for the minimum number, over all 2-colorings of $[1, n]$, of monochromatic triples that are solutions of $x + ay = z$, for a fixed integer $a \geq 2$. We will obtain this upper bound by using the Greedy Algorithm. The general idea is the same as in the previous section. We again call the colors red and blue.

First

We paint the first interval of length k red. We will have $\frac{k^2}{2a}$ monochromatic triples as solutions of $x + ay = z$.

Second

We paint the second interval blue. We want to find the length of the interval (with this color) so that the overall number of monochromatic triples is minimum.

Let the length of this interval be $(a + j)k$ (here j is the number we want to find).

The total number of monochromatic triples on the whole interval is now $\frac{k^2}{2a} + \frac{j^2k^2}{2a} = \frac{(1+j^2)k^2}{2a}$.

The total length is N $= k + (a + j)k = (1 + a + j)k$.

So the total number of monochromatic triples in terms of N is
$\frac{(1+j^2)(\frac{N}{1+a+j})^2}{2a} = \frac{(1+j^2)}{(1+a+j)^2}\frac{N^2}{2a}$.

To find the minimum, we use calculus to get $j = \frac{1}{a+1}$. The total number of monochromatic Schur triples is then $\frac{N^2}{2a(a^2+2a+2)}$.

So far so good. We have a coloring that paints the first k integers red, followed by painting the next $(a+\frac{1}{a+1})k$ integers blue.

Third

Now we try to stick red at the end of the interval, and try to lower the overall number of triples. Say the length of this interval is jk, where j is the number we want to find. The total number of monochromatic Schur triples on the whole interval is $\frac{k^2}{2a}+\frac{k^2}{2a(a+1)^2}+\frac{j^2k^2}{2a}=\frac{((a+1)^2+1+(a+1)^2j^2)k^2}{2a(a+1)^2}$.

The total length is $N=k+(a+\frac{1}{a+1})k+jk=(1+a+\frac{1}{a+1}+j)k$.

So the total number of monochromatic Schur triples in term of N is

$\frac{((a+1)^2+1+(a+1)^2j^2)}{2a(a+1)^2}(\frac{N}{(1+a+\frac{1}{a+1}+j)})^2=\frac{((a+1)^2+1+(a+1)^2j^2)N^2}{2a((a+1)^2+1+(a+1)j)^2}$.

To find the minimum, we again use calculus to get $j=\frac{1}{a+1}$. The total number of monochromatic triples is $\frac{N^2}{2a(a^2+2a+3)}$. The coloring for the whole interval is a red interval of length equal to k, a blue interval of length equal to $(a+\frac{1}{a+1})k$ and another red interval of length equal to $\frac{1}{a+1}k$. k is such that the sum of these intervals is N, i.e. $k=\frac{N}{(1+a+\frac{2}{a+1})}$.

Fourth

We try to lower the bound even further by having a blue interval of length, say, jk at the end of the previous interval. But now we get that the minimizing j is negative. So we stop.

As a conclusion, the optimal coloring is proportional to $[1,a+\frac{1}{a+1},\frac{1}{a+1}]$, with colors $[R,B,R]$ yielding that indeed the minimal number is $\frac{N^2}{2a(a^2+2a+3)}$.

7.3.2 Lower bounds

We will use a similar technique for the lower bound of the original problem. We find an upper bound for non-monochromatic triples in $[1, n]$. This gives a lower bound of the monochromatic triples.

We use the notation (R, B) and (B, R) for the non-monochromatic pair (x, y).

Definition:

Let $|R|$ be the number of red points in $[1, n]$.

Let $|B|$ be the number of blue points in $[1, n]$.

Lemma 1) $|\{(R, B), (B, R)| \;\; y > x, \;\; y - x \;$ is divisible by $a\}| \leq \frac{|R||B|}{a}$.

Proof:

Let $|r_i|$ = number of red points at position m in $[1, n]$ where $m \equiv i \bmod a$.

Let $|b_i|$ = number of blue points at position m in $[1, n]$ where $m \equiv i \bmod a$.

We remark $r_i + b_i = \frac{n}{a}$, $1 \leq i \leq a$ and $\sum_{i=1}^{a} r_i = |R|$.

$|\{(R, B), (B, R)| \;\; y > x, y - x \;$ is divisible by $a\}| - \frac{|R||B|}{a}$

$= \sum_{i=1}^{a} r_i b_i - \frac{|R||B|}{a}$

$= \sum_{i=1}^{a} r_i(\frac{n}{a} - r_i) - \frac{(\sum_{i=1}^{a} r_i)(n - \sum_{i=1}^{a} r_i)}{a}$

$= -\sum_{i=1}^{a} r_i^2 + \frac{(\sum_{i=1}^{a} r_i)^2}{a}$

≤ 0 , by the Cauchy-Schwarz inequality.

Moreover, equality holds when $r_1 = r_2 = ... = r_a$. $\square$

Let Q_a be two times the number of non-monochromatic triples of solutions of $x + ay = z$

in a 2-coloring of $[1, n]$.

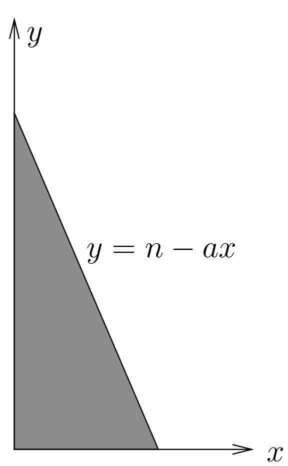
Lemma 2) $Q_a \leq \frac{|R||B|}{a} + |\{(R,B),(B,R)| \;\; y - ax \geq 0\}| + |\{(R,B),(B,R)| \;\; y + ax \leq n\}|$.

Proof:

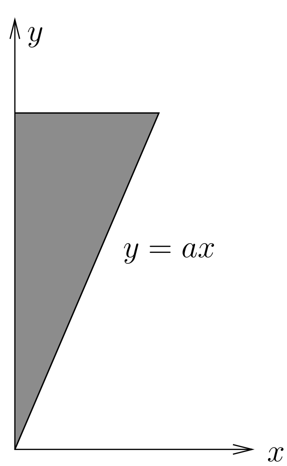
$$\begin{aligned} Q_a &= |\{\text{the non-monochromatic pair } (x,y)| \\ &\quad\quad y > x \text{ and } y - x \text{ is divisible by a}\}| \\ &\quad + |\{\text{the non-monochromatic pair } (x,y)| \;\; y - ax \geq 0\}| \\ &\quad + |\{\text{the non-monochromatic pair } (x,y)| \;\; y + ax \leq n\}| \\ &\leq \frac{|R||B|}{a} + |\{(R,B),(B,R)| \;\; y - ax \geq 0\}| + |\{(R,B),(B,R)| \;\; y + ax \leq n\}| \\ &\quad \text{, by lemma 1} \;\; \square. \end{aligned}$$

When the points on the x-axis and the y-axis are painted with either color red or blue. $|\{(R,B),(B,R)| \;\; y + ax \leq n\}|$ is the number of non-monochromatic coordinate pairs inside the triangle 1.

Similarly $|\{(R,B),(B,R)| \;\; y - ax \geq 0\}|$ is the number of non-monochromatic coordinate pairs inside the triangle 2.



Triangle 1



Triangle 2

Divide the interval $[1, n]$ into k consecutive intervals.

Let r_i be the number of red points in the interval I_i.

Let b_i be the number of blue points in the interval I_i.

Let $S_{i,j}$ be the number of non-monochromatic pairs in the square of $I_i \times I_j$.

Let $T_{i,j}$ be the number of non-monochromatic pairs in the intersection of triangle and the square $I_i \times I_j$.

Note: $r_i + b_i = \frac{n}{k}$.

Theorem 3) $Q_2 \leq \frac{57n^2}{121} + O(n)$.

We find an upper bound of Q_2 by using calculus on the equation from the previous lemma.

The main part of calculating Q_2 is to compute the maximum number of non-monochromatic pairs in triangle 1 and triangle 2 in the pictures above. We partition an interval $[1, n]$ into k smaller intervals with equal length called them $I_1, .., I_k$. However there are $I_i \times I_j$ for some i, j that intersect the triangle only partly. We denote them $T_{i,j}$.

For each $T_{i,j}$ we have two ways to bound it

1) $T_{i,j} \leq$ area of the intersection of triangle and the square $I_i \times I_j$.

2) $T_{i,j} \leq S_{i,j} = r_i b_j + r_j b_i$.

In this case, we use 11 intervals, $k = 11$.

In triangle 1, we bound $T_{1,10}, T_{2,9}, T_{2,8}, T_{3,7}, T_{3,6}, T_{5,3}, T_{5,2}$ and $T_{6,1}$ by the area of each intersecting triangle. We bound $T_{1,11}, T_{4,5}$ and $T_{4,4}$ by $S_{i,j}$.

In triangle 2, we bound $T_{2,4}, T_{3,5}, T_{3,6}$ and $T_{6,11}$ by the area of each intersecting triangle. We bound $T_{1,1}, T_{1,2}, T_{2,3}, T_{4,7}, T_{4,8}, T_{5,9}$ and $T_{5,10}$ by $S_{i,j}$.

We then run the Maple program. We get four optimal solutions to the maximum of Q_2. Two of them are

$[r_1, r_2, ..., r_{11}] = [\frac{n}{11}, \frac{n}{11}, 0, \frac{n}{11}, 0, 0, 0, 0, 0, \frac{n}{11}, 0]$ and $[\frac{n}{11}, \frac{n}{11}, 0, 0, 0, 0, 0, 0, \frac{n}{11}, \frac{n}{11}, \frac{n}{11}]$. The other two are the switching colors of the first two.

This yields the maximum of Q_2 as $\frac{57n^2}{121} + O(n)$. $\square$

Definition:

Let $M_{\chi,a}(n)$ be the number of monochromatic triples of solutions of $x + ay = z$ for a 2-coloring of $[1, n]$.

Corollary 4) $M_{\chi,2}(n) \geq \frac{7n^2}{484} + O(n)$.

Proof:

The total number of triples = |monochromatic triples| + |non-monochromatic triples|

$$= M_{\chi,a}(n) + \frac{Q_a}{2} \; .$$

Since the total number of triples is $\frac{n^2}{2a} + O(n)$, we have $M_{\chi,a}(n) \geq \frac{n^2}{2a} - \frac{Q_a}{2} + O(n)$.

The lower bound, we found, of $M_{\chi,2}(n)$ follows from the upper bound of Q_2 from Theorem 3. $\square$

Note:

1) For $a = 3$, we found, $M_{\chi,3}(n) \geq \frac{n^2}{2268} + O(n)$. We ran the calculus program on 9 intervals with a particular upper bound of $T_{i,j}$.

2) For case $a \geq 4$, we could not find a positive lower bound for $M_{\chi,a}(n)$ yet. One of the reasons is that the upper bound of $M_{\chi,a}(n)$ is very small.

7.4 The minimum number, over all r-coloring of $[1,n]$, of monochromatic Schur triples

7.4.1 A Greedy Algorithm for Upper bounds

The method to obtain the upper bounds in this section is similar to the one used in sections 2 and 3. In general we start with the first interval with color 1. Then we add interval 2 with color 2 in the optimal way. Then we add the third interval starting with color 1. If we get a positive solution , we move to the fourth interval. Otherwise we try with color 2. We keep going on in this fashion until there is no color that gives a positive solution.

Since there are many intervals involved in the computation, it is too much computation to do by hand. We wrote a computer program to help us compute the solutions for each r-coloring. The work was not so straightforward as we thought, since the details of counting triples and doing calculus at the same time are a bit tricky. But we made it at the end. We list the colorings up to $r = 5$, as examples, below. The program is available for download from the web site.

Definitions:

C = list of the coloring in order.

L = length of each interval (proportional to each other) corresponding to each color in C.

N = number of monochromatic Schur triple according to C and L.

$r = 1, \quad C = [1], \quad L = [1], \quad \text{N} = \frac{n^2}{4} + O(n).$

$r = 2, \quad C = [1,2,1], \quad L = [1, \frac{3}{2}, \frac{1}{4}], \quad \text{N} = \frac{n^2}{22} + O(n).$

$$r = 3, \quad C = [1,2,1,3,1,2,1], \quad L = [1, \tfrac{3}{2}, \tfrac{1}{4}, 3, \tfrac{1}{8}, \tfrac{487}{440}, \tfrac{47}{440}], \quad \text{N} = \frac{47n^2}{6238} + O(n) \sim \frac{n^2}{132.7234} + O(n).$$

For $r \geq 4$, the lengths of the intervals are fractions with huge numerators and denominators. So we omit C and L here.

$r = 4$, N $= \frac{696312226992930423294815 27n^2}{670769840913967048094053153 98} + O(n) \sim \frac{n^2}{963.3176} + O(n)$.
$r = 5$, N $\sim \frac{n^2}{7610.0730} + O(n)$.

For $r = 6$, the lengths of the intervals are even larger fractions. This caused Maple to slow down. We waited for about 8 hours and we stopped. We did not get an answer.

However we were not really disappointed about this failure. The algorithm is more important.

7.4.2 Lower bounds

The method used to find a lower bound in the previous two sections could not be adapted for r-colorings, $r \geq 3$. We did not make any progress for a lower bound of r-coloring cases.

7.5 About the program

$LowerBound(k, C)$

input: the number of intervals k, list of types of upper bound C of $T_{i,k-i+1}$.

output: lower bound of $M_\chi(n)$, the upper bound of Q and the optimal solution of Q.

$LowerBound2(k, C1, C2, a)$

input: the number of intervals k, list of types of upper bound $C1$ and $C2$ of $T_{i,k-i+1}$ and number a in equation $x + ay = z$.

output: lower bound of $M_{\chi,a}(n)$, the upper bound of Q_a and the optimal solution of Q_a.

$minAllST(n, r)$

input: length of intervals n, number of colors r.

output: the r-coloring of all the interval of length n that has the least number of monochromatic Schur triples.

$Ord(C, L, n)$

input: the list of coloring, the list of length corresponding to each color in C, symbol n.

output: the number of the monochromatic Schur triples of order n^2.

$Zeil(r)$

input: number of color r.

output: the coloring with length of each coloring and also the total number of triples of order n^2 obtained from the Greedy Algorithm.

7.6 Conclusion

We have new upper bounds for generalized Schur triples $x + ay = z, a \geq 2$, in the 2-coloring case. We also have new upper bounds for Schur triples $x + y = z$, for r-colorings, $r \geq 3$ that considerably improve those of [6]. But we failed to match the lower and upper bounds for these two problems. There is a possibility that other arguments in other papers [10], [3] and [9] for the lower bound used in the original problem can be adapted for the r-coloring problem. But the details of such an argument seem complicated. We believe these upper bounds are actually optimal. There might even be a beautiful simple way to solve it, but we failed to find it this time. We leave them as conjectures.

Conjectures:

1) The (asymptotic) number of minimum monochromatic triples of the form $\{x, y, x + ay\}$, $a \geq 2$ of 2-colorings of $[1, n]$, are $\frac{n^2}{2a(a^2+2a+3)} + O(n)$.

2) The (asymptotic) number of minimum Schur triples of r-colorings of $[1, n]$, $r \geq 3$, are the same as the upper bounds obtained from the Greedy Algorithm.

Chapter 8

The Symbolic Moment Calculus On Ramsey Type Problems (and how it could make YOU famous)

The expectation functional is a powerful tool in the study of combinatorial objects, and often gives us quite useful information. To find the higher moments, the computation gets complicated very fast and we need computers to do symbolic computation for us. The technique has already been demonstrated in [12], [13]. Once we find high enough moments, they could actually be useful for calculating lower bounds for enumerating combinatorial objects (see [13]).

In the first part of this chapter, we compute the first few moments of the random variable "number of monochromatic Schur Triples" defined on r-coloring of $[1, n]$, as well as those for the random variable "number of monochromatic complete graphs K_k" defined on r-edge-colorings of K_n. In the second part, we speculate about possible applications to improving the dismall known lower bounds on Ramsey numbers.

8.1 Symbolic Moment Calculus

Let $\mathcal{U}$ be a set of elements and $\mathcal{P}$ be a set of properties. For each $u \in \mathcal{U}$ and $p \in \mathcal{P}$, u either does or does not enjoy property p. Let $X : \mathcal{U} \to \mathcal{Z}_{\geq 0}$ be the random variable defined by $X(u) :=$ the number of properties in $\mathcal{P}$ enjoyed by u.

In general, we need lots of calculations in order to compute higher moments. The beauty of the present Symbolic Moment Calculus is that one can check formulas with numerical results in small cases.

8.1.1 On the Number of Monochromatic Schur Triples of r-colorings of $[1,n]$

We let $\mathcal{U}$ be set of all r-integer-colorings of $[1,n]$, and $X(u)$ be the number of monochromatic Schur triples $\{x,y,x+y\}$ of $[1,n]$ in $u \in \mathcal{U}$.

Of course, $E[X^0] = 1$, $E[X]$ is a bit harder, but higher moments get increasingly hard to do, and past the second moment, are infeasible without a computer.

Let's write

$$X = \sum_S X_S, \tag{A-1}$$

where the sum is over all triples $S, \{x,y,x+y\}$ of $[1,n]$, and X_S is the *indicator* random variable that is 1 if the triple of $[1,n]$ induced by S is monochromatic.

We also have

$$X^r = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}...\sum_{i_r=1}^{n} X_{i_1}X_{i_2}...X_{i_r}, \tag{A-2}$$

and by *linearity of expectation* and *symmetry*, we further have

$$E[X^r] = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}...\sum_{i_r=1}^{n} E[X_{i_1}X_{i_2}...X_{i_r}], \tag{A-3}$$

The difficulty in calculating the higher moments lies in how these random variables interact with each other.

The first moment:

A triple S can be written as $S = S_1 \cup S_2$ where S_1 is a triple of the form $\{x,x,2x\}$ and S_2 is a triple of the form $\{x,y,x+y\}, x \neq y$.

$E[X_{S_1}] = r(\frac{1}{r^2}) = \frac{1}{r}$.

$E[X_{S_2}] = r(\frac{1}{r^3}) = \frac{1}{r^2}$.

Already there are two formulas for $E[X]$ depending on whether the length of the interval n is odd or even.

Case 1: n is odd

$$\begin{aligned} E[X] &= \textstyle\sum_S E[X_S] \\ &= \textstyle\sum_{S_1} E[X_{S_1}] + \sum_{S_2} E[X_{S_2}] \\ &= \tfrac{1}{r}(\tfrac{n-1}{2}) + \tfrac{1}{r^2}((n-2)+(n-4)+(n-6)+...+(1)) \\ &= \tfrac{1}{r}(\tfrac{n-1}{2}) + \tfrac{1}{r^2}(\tfrac{n-1}{2})^2 \\ &= \tfrac{(n-1)(n-1+2r)}{4r^2}. \end{aligned}$$

Case 2: n is even

$$\begin{aligned} E[X] &= \textstyle\sum_{S_1} E[X_{S_1}] + \sum_{S_2} E[X_{S_2}] \\ &= \tfrac{1}{r}(\tfrac{n}{2}) + \tfrac{1}{r^2}((n-2)+(n-4)+(n-6)+...+(0)) \\ &= \tfrac{1}{r}(\tfrac{n}{2}) + \tfrac{1}{r^2}(\tfrac{n}{2})(\tfrac{n}{2}-1) \\ &= \tfrac{n(n-2+2r)}{4r^2}. \end{aligned}$$

The second moment:

The calculation for the second moment is much harder. We consider 2-tuples $[S_1, S_2]$ of triples $\{x, y, x+y\}$ of $[1, n]$.

$$E[X^2] = E[(\textstyle\sum_{S_1} X_{S_1})(\sum_{S_2} X_{S_2})] = \sum_{[S_1,S_2]} E[X_{S_1}X_{S_2}].$$

The sum depends on how S_1 and S_2 interact with each other. For each configuration of $S_1 \cup S_2, 2 \le |S_1 \cup S_2| \le 6$. For example, $|S_1 \cup S_2| = 6$ when $|S_1| = |S_2| = 3$ and these two triples do not intersect.

Let $K := S_1 \cup S_2$ represent each isomorphic configuration. While the first moment has only 2 isomorphic configurations $\{x, x, 2x\}$ and $\{x, y, x+y\}, \quad x \neq y$, the second moment has 42 configurations.

We call the number of each isomorphic K occurring in the sum the *weight* $W(K)$. We need to find $E[X_{S_1}X_{S_2}]$ and the weight of each K. Then apply the relation:

$$E[X^2] = \sum_{[S_1,S_2]} E[X_{S_1}X_{S_2}] = \sum_{K\in\mathcal{K}} E[X_{S_1}X_{S_2}]W(K). \qquad \text{(A-4)}$$

The first quantity $E[X_{S_1}X_{S_2}]$ is not hard to compute.
Let $p := |S_1 \cup S_2|$.
$E[X_{S_1}X_{S_2}] = \frac{r}{r^p}$ if $|S_1 \cap S_2| \neq 0$ and $E[X_{S_1}X_{S_2}] = \frac{r^2}{r^p}$ if $|S_1 \cap S_2| = 0$.
To compute $W(K)$ is harder. One way to do so is to use Goulden-Jackson Cluster method, see [7]. This method gives us a generating function in terms of n.

Formulas for $E[X^2]$:

$E[X^2]$ has 12 formulas up to the values of n mod 12.

I obtain the formulas by using both the Goulden-Jackson Cluster method and *polynomial ansatz*. For each K contributes to $E[X^2]$, I first find the generating function using Goulden-Jackson Cluster method. This gives me the period l of K. Then I numerically compute the value of $W(K)$ for each n. Finally I interpolate the polynomial, $p(n)$ of degree at most 4 with the period l that fits the numerical results. This polynomial ansatz gives a rigorous proof of the second moment.

As an example, we show the formula of $E[X^2]$ in the case when $n \equiv 1$ mod 12.

$E[X^2] = \frac{(n-1)(24r^3-76r-27n+65-9n^2+12rn^2+24r^2n+3n^3-16r^2)}{48r^4}$

The complete solutions can be found at the author's website.

The higher moment, the m-th moment:

We now consider m-tuples $[S_1, S_2, ..., S_m]$ of triples $\{x, y, x+y\}$ of $[1, n]$.

$E[X^m] = \sum_{[S_1,S_2,..,S_m]} E[X_{S_1} X_{S_2} .. X_{S_m}]$.

We again consider the isomorphic configuration K, its weight $W(K)$ and its probability $E[X_{S_1} X_{S_2} .. X_{S_m}]$.

For $E[X^3]$, we find out that there are more than 500 isomorphic configurations of $S_1 \cup S_2 \cup S_3$ with at least 72 formulas up to the values of n mod 72. I tried for about 3 weeks, but in the end, I felt like it was too much effort. We were not lucky with computing $E[X^3]$ but at least we know it is *not easy* to compute higher moments.

8.1.2 On the Number of Monochromatic K_k on K_n

Here, we let $\mathcal{U}$ be set of all r-edge-colorings of K_n, and $X(u)$ be the number of monochromatic K_k in $u \in \mathcal{U}$.

It is clear that $E[X^0] = 1$, $E[X] = \frac{r}{r^{\binom{k}{2}}} \binom{n}{k}$. However, computing $E[X^2]$ is harder. We need a computer for $E[X^3]$.

We are interested in the formulas of $E[X^s], s \geq 0$ where the inputs are numeric m and k for the m^{th}-moment and the monochromatic K_k, and the output is in the form of symbolic n and r for K_n and r-edge-coloring.

Write

$$X = \sum_S X_S$$

where the sum is over all k-subsets of $\{1..n\}$, and X_S is the *indicator* random variable that is 1 if the subgraph of K_n induced by S is monochromatic.

The first moment:

By simple calculation, we can show $E[X_S] = \frac{r}{r^{\binom{k}{2}}}$. Thus, we find the expectation

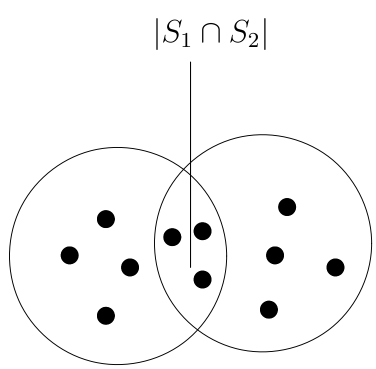
$$E[X] = \frac{r}{r^{\binom{k}{2}}} \binom{n}{k}. \tag{A-5}$$

The second moment:

We again start with the relation,

$E[X^2] = \sum_{[S_1,S_2]} E[X_{S_1} X_{S_2}]$.

The isomorphic configuration K is entirely up to the size of $S_1 \cup S_2$.



The goal is to figure out $E[X_{S_1} X_{S_2}]$ and the weight of each K.

$E[X_{S_1} X_{S_2}]$ can be found as follow.

Let $j = |S_1 \cap S_2|$. If $j = 0$ or 1 then $E[X_{S_1} X_{S_2}] = (\frac{r}{r^{\binom{k}{2}}})^2$. If $j = |S_1 \cap S_2| \geq 2$ then all the edges of $S_1 \cup S_2$ must have the same color and hence $E[X_{S_1} X_{S_2}] = \frac{r}{r^{2\binom{k}{2} - \binom{j}{2}}}$. In this case, the weight W is also not hard to compute.

$W = \frac{n!}{(k-j)!(k-j)!j!(n-((k-j)+(k-j)+j))!}$ where $j = |S_1 \cap S_2|$.

We then make use of the previous formula for $E[X^2]$.

$$E[X^2] = \sum_{[S_1,S_2]} E[X_{S_1}X_{S_2}] = \sum_{j=0}^{k} Prob(K)W(K) \tag{A-6}$$

where j represents $|S_1 \cap S_2|$ and $Prob(K) = E[X_{S_1}X_{S_2}]$ where $K = S_1 \cup S_2$.

The equation (A-6) is not so complicated, we can write it out explicitly.

$E[X^2] = \sum_{j=0}^{1}(\frac{r}{r^{\binom{k}{2}}})^2 T(j) + \sum_{j=2}^{k} \frac{r}{r^{2\binom{k}{2}-\binom{j}{2}}} T(j)$,

where $T(j) = \frac{n!}{(k-j)!(k-j)!j!(n-((k-j)+(k-j)+j))!}$.

For example when $k = 3$, $E[X^2]$ can be written as

$E[X^2] = \frac{n!}{3!3!0!(n-6)!}\frac{1}{r^4} + \frac{n!}{2!2!1!(n-5)!}\frac{1}{r^4} + \frac{n!}{1!1!2!(n-4)!}\frac{1}{r^4} + \frac{n!}{0!0!3!(n-3)!}\frac{1}{r^2}$

$= \frac{(n-2)(n-1)n(n^3-3n^2+2n-6+6r^2)}{36r^4}$

For $k = 4$,

$E[X^2] = \frac{n!}{4!4!0!(n-8)!}\frac{1}{r^{10}} + \frac{n!}{3!3!1!(n-7)!}\frac{1}{r^{10}} + \frac{n!}{2!2!2!(n-6)!}\frac{1}{r^{10}} + \frac{n!}{1!1!3!(n-5)!}\frac{1}{r^8} + \frac{n!}{0!0!4!(n-4)!}\frac{1}{r^5}$

$= \frac{(n-3)(n-2)(n-1)n(n^4-6n^3+11n^2-102n+360+96r^2n-384r^2+24r^5)}{576r^{10}}$

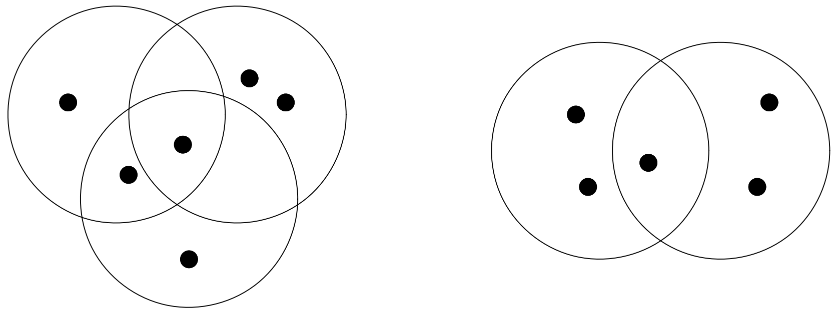
For the general case, the m^{th} moment:

We now consider m-tuples $[S_1, S_2, ..., S_m]$ of k subsets of $\{1, .., n\}$.

$E[X^m] = E[(\sum_{S_1} X_{S_1})(\sum_{S_2} X_{S_2})...(\sum_{S_m} X_{S_m})] = \sum_{[S_1,S_2,..,S_m]} E[X_{S_1}X_{S_2}..X_{S_m}]$.

We again consider the isomorphism class of the configuration K, its probability, $E[X_{S_1}X_{S_2}..X_{S_m}]$, and its weight $W(K)$.

The weight $W(K)$ can be computed directly from the way $S_1, S_2, ..., S_m$ intersect each other. The probability can be computed in a similar way.



$prob(K) = \prod_D \frac{r}{r^{e(d)}}$ where we multiply through all the chains d in D and $e(d)$ is the number of edges in d. S_i is in the chain d if there is some S_j in d such that $|S_i \cap S_j| \geq 2$.

Example

From the picture above,

$W(K) = \binom{n}{1!,1!,1!,1!,2!,1!,2!,2!,(n-11)!}$

Since there are three chains, the probability is

$Prob(K) = E[X_{S_1}X_{S_2}X_{S_3}] = (\frac{r}{r^8})(\frac{r}{r^3})(\frac{r}{r^3})$. $\square$

Although we know how to compute $W(K)$ and $prob(K)$ for each K, there are still lots of isomorphic configurations when the number of moment is high.

We show the computation of third moment when the input is k.

$$E[X^3] = \sum_{i_3=0}^{k-j}\sum_{i_2=0}^{k-j}\sum_{i_1=0}^{k-j}\sum_{j=0}^{k} W(K)prob(K) \tag{A-7}$$

where $W(K) = \frac{n!}{j!i_1!i_2!i_3!(k-j-i_1)!(k-j-i_2)!(k-j-i_3)!(n-|S_1 \cup S_2 \cup S_3|)!}$,

$prob(K)$ depends how S_1, S_2 and S_3 intersect.

The number of sums grows exponentially as the number of moment goes up. We need symbolic computation program to compute the higher moments for us. I implemented this idea in Maple program. Disappointingly, even with a computer we could compute up to only the fifth moment. The program and the outputs are also at the author's website.

The formulas may not have detectable patterns in either k or m. However we have a nice way to think about how these formulas arise.

8.2 Applications

8.2.1 Introduction

Every student of Ramsey theory has heard the following statement: *In any party of six people, there is either a group of three people who are mutual strangers or a group of three people who are mutual acquaintances (or both)*

We define $R(k, k)$ to be the smallest number n for which for any 2-coloring of the edges of K_n, say with red or blue, contains either a totally red K_k or a totally blue K_k. It is known that $R(3, 3) = 6$ and $R(4, 4) = 18$, but $R(5, 5)$ is still unknown. We see that the exact value of $R(k, k)$ is very hard to determine because of the gigantic possibilities of edge-colorings.

The question about the asymptotic behavior of $R(k, k)$ is a famous open problem in combinatorics. There is even a monetary prize, \$250, for the answer of the problem. Paul Erdös used the first moment $E[X]$ to obtain a lower bound of $\lim_{k\to\infty} R(k, k)$ (if it exists). We can also state the idea in terms of the pigeonhole principle.

If $E[X] < 1$ then $P(X = 0) > 0$.

In the language of the Moment Calculus ,with the same notation as above, we want to find n such that $E[X] < 1$ implies $R(k, k) > n$.

However the full strength of the idea along the same lines can be obtained by the generalization of the Principle of Inclusion-Exclusion (PIE), called *Bonferroni's inequality.*

Recall that the Principle of Inclusion-Exclusion can be stated as follows.

$$P(X=0) = \sum_{s\geq 0}(-1)^s E[\binom{X}{S}]. \tag{A-1}$$

To compute $P(X=0)$ from (A-1) is definitely out of reach. But we can still hope to get useful information from Bonferroni's inequality, that who knows?, may lead to an improvement of the lower bound.

Bonferroni's inequality:

For any odd m,

$$P(X=0) \geq \sum_{s=0}^{m}(-1)^s E[\binom{X}{S}], \tag{A-2}$$

For any even m,

$$P(X=0) \leq \sum_{s=0}^{m}(-1)^s E[\binom{X}{S}]. \tag{A-3}$$

Erdös used $m=1$ to get the lower bound $\lim_{k\to\infty} R(k,k)^{\frac{1}{k}} \geq \sqrt{2}$. However $m=3$, and even $m=5$, do not seem to improve the lower bound at all. In my opinion, we need a much bigger m. That means we also need lots more computation. I will discuss my idea about how to do the calculation with a lot of moments in the next section. Even if the naive Bonferroni's sieve would fail to do the job, the insight gained may be useful in using other, more delicate future sieves.

8.2.2 Calculation

We start this section by crossing our fingers and really believing in the beauty of Mathematics. The plan is clear. We will use a lot of higher moments as an input to Bonferroni's inequality to try to improve the lower bound of $\lim_{k\to\infty} R(k,k)^{\frac{1}{k}}$.

But how many moments do we need to improve the famous lower bound? We are not sure yet. Let's say we use $m = k$ to begin with.

There are lots of terms when the moments goes up. We can not calculate every term. However we will pick some significant terms in each of the moments and pick a value of n in terms of k with the hope that the rest of terms will be dominated and at the same time n improves the lower bound. In other words, we only aim at the first leading terms in the *asymptotic expansion* for the moments.

We start our calculation with the term which we think is the biggest one. The term in the moment $E[X^i]$ where all S_i completely intersect; $|S_1 \cup S_2 \cup ... \cup S_m| = k$. We call this term A_1. Let $W_j(i)$ be the number of A_j in $E[X^i]$. We want to find $W_j(i)$ of each A_j.

For A_1, we have that,

$$A_1 = E[X] = \frac{2}{2^{\binom{k}{2}}} \frac{(n)_k}{k!}. \tag{A-4}$$

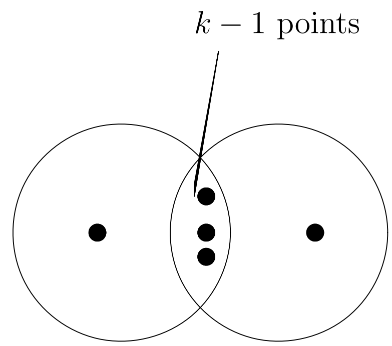
where $(n)_k = n(n-1)(n-2)...(n-k+1)$. We call $(n)_k$ a *falling factor.*

Since there is only one way to get the term A_1 for each of the moment $E[X^i], \ \ i \geq 1$, $W_1(i) = 1$ for $i \geq 1$.

Now we consider the second biggest term where $|S_1 \cup S_2 \cup .. \cup S_m| = k+1$.
There are many ways to get this configuration. We first consider the one with most intersection points.

First: Consider the term A_{22} where $|S_1 \cap S_2 \cap .. \cap S_m| = k-1$.

Let P_i be the set of number of moments that the point i contains. We see $P_i = \{1, 2, ..., m\}, \ \ 1 \leq i \leq k-1$. However the last two points, $P_k \cup P_{k+1} = \{1, 2, 3, ..., m\}$. where $P_k \neq \{ \ \ \}$ and $P_{k+1} \neq \{ \ \ \}$.



The number of edges in $|S_1 \cup S_2 \cup .. \cup S_m| = \binom{k-1}{2} + 2(k-1)$

$$= \frac{k^2}{2} - \frac{3k}{2} + 1 + 2k - 2$$

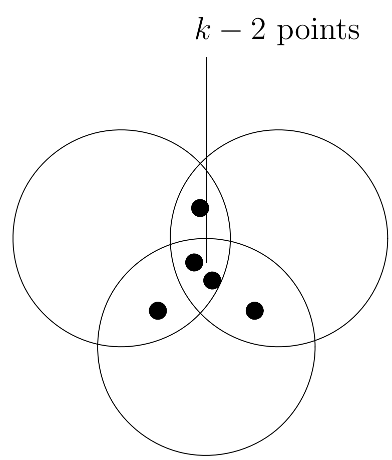
$$= \frac{k^2}{2} + \frac{k}{2} - 1.$$

$$A_{22} = \frac{2}{2^{\frac{k^2}{2}+\frac{k}{2}-1}} \frac{(n)_{k+1}}{(k-1)!1!1!}. \tag{A-5}$$

By elementary counting we know, $W_{22}(i) = \frac{2^i}{2} - 1; \quad i \geq 2$.

Note: $A_1/A_{22} = (\frac{2}{2^{\binom{k}{2}}} \frac{(n)_k}{k!})/(\frac{2}{2^{\frac{k^2}{2}+\frac{k}{2}-1}} \frac{(n)_{k+1}}{(k-1)!1!1!}) = \frac{2^k}{2} \frac{1}{(n-k-1)k}$.

Second: Consider the term A_{23} where $|S_1 \cap S_2 \cap .. \cap S_m| = k-2$.



Each of the $k-2$ points contains in each of $S_1, S_2, ..., S_m$. However the last three points need to contain two copies of $\{1, 2, ..., m\}$ where the number appears in each point at most once.

$$\begin{aligned}\text{The number of edges in } |S_1 \cup S_2 \cup .. \cup S_m| &= \tbinom{k-2}{2} + 3(k-2) + 3 \\ &= \tfrac{k^2}{2} - \tfrac{5k}{2} + 3 + 3k - 3 \\ &= \tfrac{k^2}{2} + \tfrac{k}{2}.\end{aligned}$$

Hence,

$$A_{23} = \frac{2}{2^{\frac{k^2}{2}+\frac{k}{2}}}\left(\frac{(n)_{k+1}}{(k-2)!1!1!1!}\right). \tag{A-6}$$

And,

$W_{22}(i) = S(i, 3) = \frac{1}{2} - \frac{2^i}{2!} + \frac{3^i}{3!}; \quad i \geq 3$ where $S(i, l)$ is the Stirling number of second kind.

Note: $A_1/A_{23} = (\frac{2}{2^{\binom{k}{2}}}\frac{(n)_k}{k!})/\frac{2}{2^{\frac{k^2}{2}+\frac{k}{2}}}\frac{(n)_{k+1}}{(k-2)!1!1!1!} = 2^k \frac{1}{(n-k-1)k(k-1)}$.

In general:

We consider the case where $|S_1 \cup S_2 \cup .. \cup S_m| = k + l$ and $|S_1 \cap S_2 \cap .. \cap S_m| = k - l, \;\; 1 \leq l \leq k-1$. We have that $P_i = \{1, 2, ..., m\}, \;\; 1 \leq i \leq k-l$. However the last $l+1$ points contain l copies of $\{1, 2, 3, ..., m\}$ where the number appears in each point at most once.

$$\begin{aligned}\text{The number of edges in } |S_1 \cup S_2 \cup .. \cup S_m| &= \tbinom{k+1}{2} \\ &= \tfrac{k^2}{2} + \tfrac{k}{2}.\end{aligned}$$

Hence,

$$A_{2(l+1)} = \frac{2}{2^{\frac{k^2}{2}+\frac{k}{2}}}(\frac{(n)_{k+1}}{(k-l)!1!1!1!}). \tag{A-7}$$

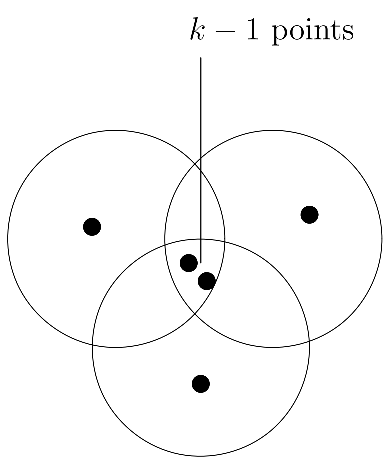
We can write out $W_{2(l+1)}(i)$ by using the summation form of Stirling numbers of second kind.

$S(i, l+1) = \frac{1}{(l+1)!}\sum_{j=0}^{l+1}(-1)^{(l+1-j)}\binom{l+1}{j}j^n; \quad i \geq l+1$

Note: $A_1/A_{2(l+1)} = 2^k \frac{1}{(n-k-1)(k)_l}$.

So far so good. We have everything in control. Now we consider the third biggest term where $|S_1 \cup S_2 \cup .. \cup S_m| = k+2$. At this point the difficulty starts since counting the number of edges in $|S_1 \cup S_2 \cup .. \cup S_m|$ and the weight W is getting more complicated.

First: Consider the term A_{33} where $|S_1 \cap S_2 \cap .. \cap S_m| = k-1$.



By performing a routine calculation, we have that

The number of edges in $|S_1 \cup S_2 \cup .. \cup S_m| = \binom{k-1}{2} + 3(k-1)$

$= \frac{k^2}{2} + \frac{3k}{2} - 2.$

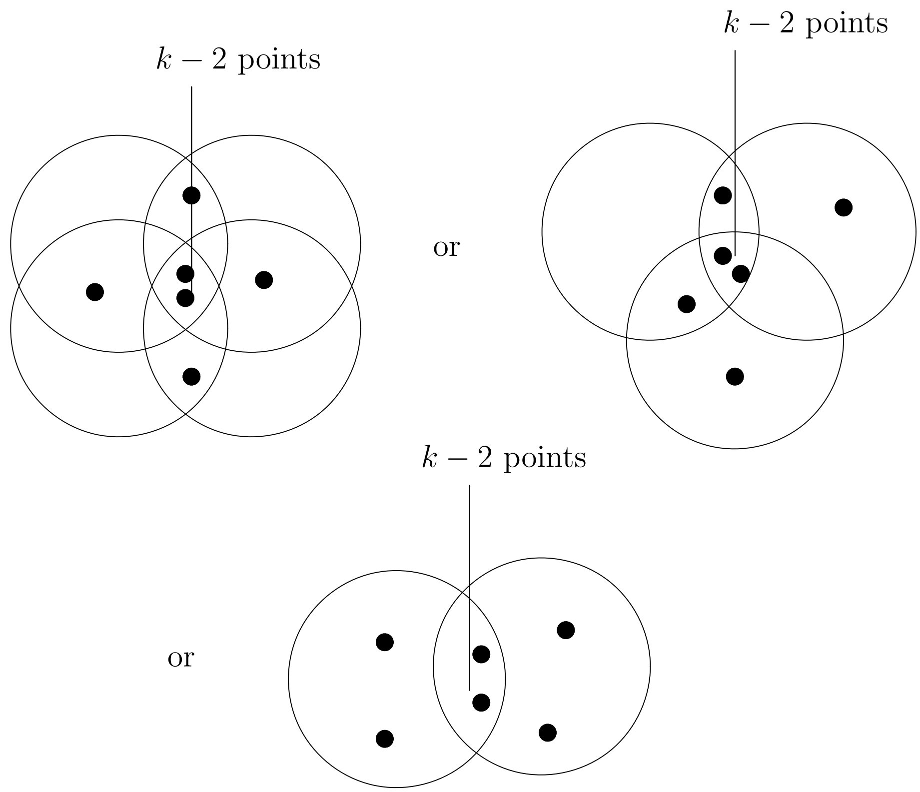
$$A_{33} = \frac{2}{2^{\frac{k^2}{2}+\frac{3k}{2}-2}}(\frac{(n)_{k+2}}{(k-1)!1!1!1!}). \tag{A-8}$$

And,

$W_{33}(i) = S(i,3), \quad i \geq 3.$

Note: $A_1/A_{33} = \frac{2^{2k}}{4}\frac{1}{(n-k-1)(n-k-2)(k)}$.

Second: Consider the term A_{34} where $|S_1 \cap S_2 \cap .. \cap S_m| = k-2$.



There are more subcases to consider in this case. Some of the subcases are harder to count the weights of. Even though the problem looks like a generalized problem of Stirling numbers of second kind. We got stuck here. I hope to use symbolic computation to help make progress in the future.

Bonferroni's inequality Calculation

Recall that we will use Bonferroni's inequality (A-2) in section 8.2.1 to try to improve the known lower bound.

Since we already calculated some of the very first terms, let's try to see the impact of these terms on Bonferroni's inequality. We write the right hand side of equation (A-2) as

$\sum_{s=0}^{m}(-1)^s E[\binom{X}{S}] = 1 + c_1 A_1 + c_{22}A_{22} + c_{23}A_{23} + ... + c_{2k}A_{2k} + ss.$

where ss represents other terms that we are not able to calculate yet.

Lemma 5. $c_1 = -1, c_{2l} = \frac{(-1)^l}{l!}$, *if we use* $m \geq k$ *in the above equation.*

Proof. $c_1 = -1$ since only $E[X]$ contributes 1 to this term.

$c_{22} = \frac{1}{2}$,

since $S(i,2) = \frac{2^i}{2} - 1, \quad i \geq 2$ so $E[\binom{X}{k}], \quad k \geq 3$ contributes 0 to this term and $E[\binom{X}{2}]$ contributes $\frac{1}{2}$ to this term.

$c_{23} = -\frac{1}{6}$,

since $S(i,3) = \frac{1}{2} - \frac{2^i}{2!} + \frac{3^i}{3!}, \quad i \geq 3$ so $E[\binom{X}{k}], \quad k \geq 4$ contributes 0 to this term and $E[\binom{X}{3}]$ contributes $\frac{1}{6}$ to this term.

The remaining terms work similarly. □

Now, let's see how these parts affect the calculation for the lower bound:

$\sum_{s=0}^{k}(-1)^s E[\binom{X}{S}]$

$= 1 - A_1 + \frac{1}{2}A_{22} - \frac{1}{6}A_{23} + ... + \frac{(-1)^k}{k!}A_{2k} + ss.$

$$= 1 - A_1 + \frac{1}{2}(\frac{2nk}{2^k})A_1 - \frac{1}{6}(\frac{nk(k-1)}{2^k})A_1 + \frac{1}{24}(\frac{nk(k-1)(k-2)}{2^k})A_1 - ... + \frac{(-1)^k}{k!}A_{2k} + ss.$$

$$= 1 - A_1 + \frac{1}{2}(\frac{nk}{2^k})A_1 + \frac{n}{2^k}A_1(\frac{k}{2!} - \frac{k(k-1)}{3!} + \frac{k(k-1)(k-2)}{4!} - ... + \frac{(-1)^k}{k!}k!) + ss.$$

$$= 1 - A_1 + \frac{1}{2}(\frac{nk}{2^k})A_1 + \frac{n}{2^k}A_1(\sum_{i=2}^{k}\frac{k!(-1)^i}{i!(k-i+1)!}) + ss.$$

There might be an easy interpretation for the sum on the right hand side that I have overlooked. But one easy way to solve it is to plug into a Maple program.

$$= 1 - A_1 + \frac{1}{2}(\frac{nk}{2^k})A_1 + \frac{n}{2^k}A_1(\frac{(-1)^k+k}{k+1}) + ss.$$

$$= 1 - A_1 + \frac{n}{2^k}A_1(\frac{k}{2} + \frac{(-1)^k+k}{k+1}) + ss.$$

This is the start of the calculation. We did not make much progress. But I am excited to share the ideas here. In future we hope to calculate more of the terms in ss. We hope to see patterns or to find the way to show that the other terms are small.

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