The Kahn–Kalai Conjecture in random graph theory

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1 Expected number of edges in $\mathcal{G}_{n,p}$

Let X be the random variable of number of edges in $G_{n,p}$.

If $p = \frac{1}{10}$ then $E[X] = \binom{n}{2}\frac{1}{10}$.

On the other hand, find p (as a function of n) such that $E[X] = 1$.

Answer: $p = \frac{2}{n^2}$.

If $p \ll \frac{1}{n^2}$ then $E[X] \to 0, \quad \text{as } n \to \infty$.
If $p \gg \frac{1}{n^2}$ then $E[X] \to \infty, \quad \text{as } n \to \infty$.

That is the expectation threshold, $p_E(\text{number of edges})$ is $\frac{1}{n^2}$.

2 $\mathcal{G}_{n,p}$ contains a copy of a subgraph H whp for any positive constant p

Assume a fixed subgraph H has k vertices and e edges.

Let X be the random variable of number of copies of H in $\mathcal{G}_{n,p}$.

We want to show $P(X \geq 1) \to 1$ as $n \to \infty$. This is equivalent to show that $P(X = 0) \to 0$ as $n \to \infty$.

Let A_i, $1 \leq i \leq \binom{n}{k}$ be the event where the chosen k vertices from $\mathcal{G}_{n,p}$ contains no induced subgraph H.

Partition vertices $[n]$ into a disjoint subset $U_1, U_2, \ldots U_{n/k}$ each of which contains k vertices. Note that $P(A_{U_i})$ are all the same and $P(A_{U_i}) \leq 1 - p^e$.

$$P(X = 0) = P(A_1 \cap A_2 \cap \cdots \cap A_{\binom{n}{k}}) \leq P(A_{U_1} \cap A_{U_2} \cap \cdots \cap A_{U_{n/k}}) = P(A_{U_1})^{n/k} \to 0,$$

as $n \to \infty$.

3 Graph Evolution

In a 1960 paper, Erdős and Rényi described the behavior of $\mathcal{G}(n,p)$ very precisely for various values of p. Their results included that:

- If $np < 1$, then a graph in $\mathcal{G}(n,p)$ will almost surely have no connected components of size larger than $O(\log(n))$.
- If $np = 1$, then a graph in $\mathcal{G}(n,p)$ will almost surely have a largest component whose size is of order $n^{2/3}$.
- If $np \to c > 1$, where c is a constant, then a graph in $\mathcal{G}(n,p)$ will almost surely have a unique giant component containing a positive fraction of the vertices. No other component will contain more than $O(\log(n))$ vertices.
- If $p < \frac{(1-\varepsilon)\ln n}{n}$, then a graph in $\mathcal{G}(n,p)$ will almost surely contain isolated vertices, and thus be disconnected.
- If $p > \frac{(1+\varepsilon)\ln n}{n}$, then a graph in $\mathcal{G}(n,p)$ will almost surely be connected.

4 Example of Threshold function

Let $\mathcal{F}$: contain a copy of a triangle. To show $p_c(\mathcal{F}) \asymp \frac{1}{n}$.

Let X be a random variable of number of copies of a triangle in $\mathcal{G}_{n,p}$.

First, we show that, given $p \ll \frac{1}{n}$, $P(X = 0) \to 1$, as $n \to \infty$.

$$E[X] = \binom{n}{3} p^3 \approx \frac{n^3}{6} p^3 \to 0.$$

The second moment method

Second, we show that, given $p \gg \frac{1}{n}$, $P(X \geq 1) \to 1$, as $n \to \infty$.

This is a difficult part as $E[X] \to \infty$ does not imply $P(X \geq 1) \to 1$. $E[X]$ may be large simply because X is very large for just a few values. So X may still be 0 for most of $G \in \mathcal{G}_{n,p}$. We have to resource to the *second moment method.*

Theorem 1. *Let $X_n \geq 0$ be an integer valued random variable. If $E[X_n] > 0$ for n large and $\frac{Var(X_n)}{E[X_n]^2} \to 0$ then $X_n > 0$ whp.*

Proof. By Chebyshev's inequality $P(|X - E[X]| \geq E[X]) \leq \frac{Var(X)}{E[X]^2}$, from where the result follows. □

In this problem,

$$\begin{aligned}
E[X^2] &= \binom{n}{3,3,n-6}p^6 + \binom{n}{1,2,2,n-5}p^6 + \binom{n}{2,1,1,n-4}p^5 + \binom{n}{3,n-3}p^3 \\
&\approx \frac{n^6p^6}{3!3!} + \frac{n^5p^6}{2!2!} + \frac{n^4p^5}{2!} + \frac{n^3p^3}{3!} \\
&= \frac{n^6p^6}{3!3!} + \text{ smaller terms }, \quad \text{as } np \gg 1.
\end{aligned}$$

We then see

$$\frac{Var(X)}{E[X]^2} = \frac{E[X^2] - E[X]^2}{E[X]^2} = \frac{E[X^2]}{E[X]^2} - 1 \to 0,$$

as $E[X] \approx \dfrac{n^3p^3}{3!}$.

5 Expected number of isolated vertices in $\mathcal{G}_{n,p}$

Let the random variable

$$X_i = \begin{cases} 1, & \text{if vertex } i \text{ is an isolated vertex} \\ 0, & \text{otherwise.} \end{cases}.$$

Notice that $P(X_i = 1) = (1-p)^{n-1}$ and $P(X_i = 0) = 1 - (1-p)^{n-1}$

Let $X = X_1 + X_2 + \cdots + X_n$.

Then $E[X] = E[\sum_{i=1}^{n} X_i] = n \cdot (1-p)^{n-1}$.

With simple calculation, we show that

$$p_E(\text{number of isolated vertices}) \asymp \frac{\ln n}{n}.$$

References

[1] Reinhard Diestel *Graph Theory*, Springer, 3rd edition.

[2] Jinyoung Park, Huy Tuan Pham, *A Proof of the Kahn-Kalai Conjecture*, arXiv:2203.17207v2